The Legacy of Leonhard Euler

A Tricentennial Tribute

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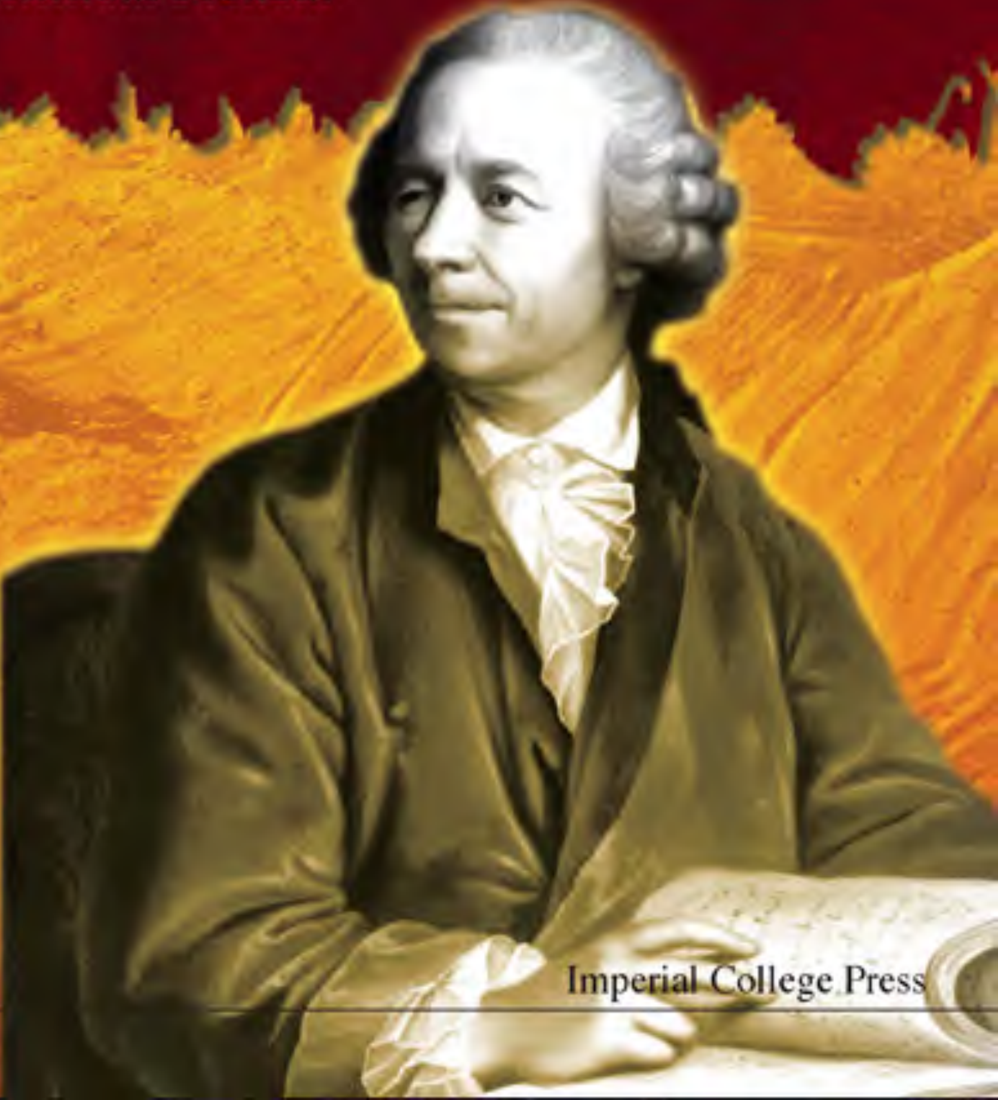
Lokenath Debnath



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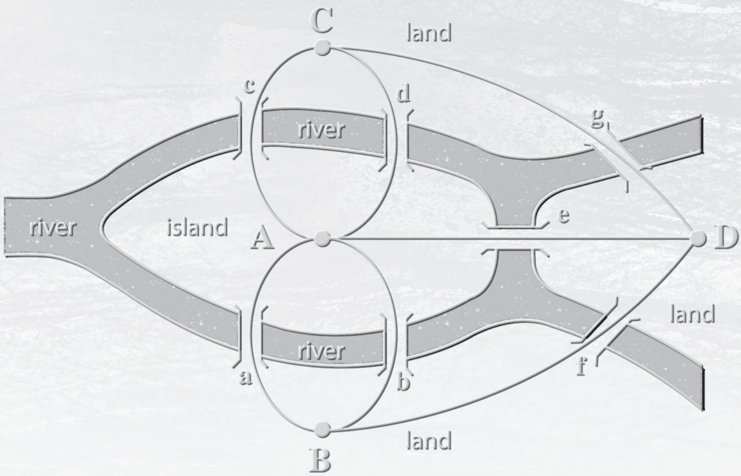

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A Tricentennial Tribute

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Leonhard Euler (1707–1783)

To my wife **Sadhana**, grandson **Kirin**, and
granddaughter **Princess Maya**,
with love and affection.

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Preface

Leonhard Euler (1707-1783) was a universal genius and one of the most brilliant intellects of all time. He made numerous major contributions to eighteenth century pure and applied mathematics, solid and fluid mechanics, astronomy, physics, ballistics, celestial mechanics and optics. Among the greatest mathematical and physical scientists of all time including Newton, Leibniz, Gauss, Riemann, Hilbert, Poincaré, and Einstein, Euler's monumental contributions are generally considered unique and fundamental and have shaped much of the modern mathematical sciences. The Eulerian universal view is the dominant influence in the fields of physics, astronomy, continuum mechanics, natural philosophy, pure and applied mathematics. He published almost 900 original research papers, memoirs, and 25 books and treatises on mathematical and physical sciences. Even without the publication of his collected works, *Leonhardi Euleri Opera Omnia*, still in the process of being edited by the Swiss Academy of Sciences, his voluminous published works clearly demonstrate his amazing creativity, achievements and contributions to a wide variety of subjects in mathematical, physical, and engineering sciences. He also made contributions to other disciplines including geography, chemistry, cartography, music, history and philosophy of science.

The following quotations give some idea of the special veneration and affection in which he was held by his contemporaries and successors. P. S. Laplace wrote: "Read Euler, read Euler, he is the master of us all." It is a delight to quote Karl Friedrich Gauss: "... the study of Euler's works will remain the best school for different fields of mathematics and nothing else can replace it." On the other hand, the great twentieth century mathematician André Weil said: "No mathematician ever attained such a position of undisputed leadership in all branches of mathematics, pure and applied, as Euler did for the best part of the eighteenth century."

The tercentenary of Euler's birth has recently been celebrated with glorious success to pay a special tribute to this legendary mathematical and physical scientist of the eighteenth century. There is absolutely no doubt that Euler laid the solid foundations on which his contemporaries and successors of the last three centuries were able to build new ideas, results, theorems and proofs. His extraordinary genius created a simple language and style, unique symbols, and notations in which mathematical and physical sciences have developed ever since. His name is also synoymously associated with a large number of results, terms, equations, theorems, and proofs in mathematics and science.

Throughout his extensive research contributions and lucid writings, Euler was always influenced by his own thought as follows: "Since a general solution must be judged impossible from want of analysis, we must be content with the knowledge of some special cases, and that all the more, since the development of various cases seems to be the only way to bringing us at last to a more perfect knowledge." In addition, Euler's quest of new knowledge was simple and direct. His standards of mathematical rigor were far more primitive than those of today, but as Richard Feynman (1918-1988), an American genius, so cogently observed in the twentieth century: "... However, the emphasis should be somewhat more on how to do the mathematics quickly and easily, and what formulas are true, rather than the mathematician's interest in methods of rigorous proof." Euler has often been criticized for his lack of mathematical clarity, elegance and rigor. Intuition played an important role in his discoveries. He was always interested in creating a set of new ideas and results in the most diverse fields of mathematical and physical sciences. So, it is perhaps true that Euler's work met all requirements for rigor in his time. He was often satisfied when his intuition gave him full confidence that the proof of results could be carried through to complete mathematical rigor and then assigned the completion of the proof to others.

In pure mathematics, his major research fields included differential and integral calculus, infinite series and products, algebra, number theory, geometry of curves and surfaces, topology, graph theory, ordinary and partial differential equations, calculus of variations, special functions, elliptic functions, and integrals. In applied mathematics, he published papers on the mechanics of particles and of solid bodies, elasticity and fluid mechanics, optics, astronomy, lunar, and planetary motion. He also wrote many textbooks on mechanics, mathematical analysis, algebra, analytic geometry, differential geometry, and the calculus of variations. In mathematical physics, Euler discovered the fundamental partial differential equations for the motion of inviscid incompressible and compressible fluid flows, and applied them to the blood flow in the human body. In the theory of heat, he closely followed Daniel Bernoulli to describe heat as an oscillation of molecules. He mathematically investigated the propagation of sound waves and obtained many original results on refraction and dispersion of light. Euler was one of the few scientists of the eighteenth century to favor the wave theory as opposed to the particle theory of light. Euler also made remarkable contributions to applied mathematics and engineering science. For example, he studied the bending of beams and calculated the critical load of columns. He described the perturbation effect of celestial bodies on the orbits of planets. He obtained the paths of projectiles in a resisting medium. He worked on the theory of tides and currents. His study on the design of ships helped navigation. His three volumes on achromatic optical instruments contributed to the design of microscopes and telescopes.

Euler maintained extensive contacts and correspondence with many of the most eminent mathematical scientists of the time including Christian Goldbach, A. C. Clairaut, Jean d'Alembert, Joseph Louis Lagrange, and Pierre Simon Laplace. This led to the development of personal and professional relationship between them. There was an amicable correspondence between Euler and Goldbach, and Euler and Clairaut which dealt with topical problems of number theory, mathematical analysis, differential equations, fluid mechanics, and celestial mechanics. There were neither any disagreements nor claims of one against the other. They discussed all mathematical ideas and problems openly, often significantly prior to their publication. Euler in Berlin and d'Alembert in Paris had an extensive mathematical correspondence over many years. In 1757, they had a strong disagreement, which eventually led to an estrangement, on whether discontinuous or non-differentiable functions are admissible solutions of the vibrating string problem. There was also a priority dispute between them on the theory of the precession of the equinoxes and nutation of the axis of the Earth. However, after d'Alembert visited Euler in Berlin in 1763, their relation became more cordial. In 1759, the young Lagrange joined in the discussion of solutions with a controversial article which was criticized by both Euler and d'Alembert. However, Lagrange sided with most of Euler's views. In 1761, Lagrange, seeking to meet the criticisms of d'Alembert and others, provided a different treatment of the vibrating string problem. The debate continued for another twenty years with no resolution. The issues in dispute were not resolved until Joseph Fourier picked up the subject in the next century. Although Euler made an important and seminal contribution to calculus of variations, Lagrange, at the age of 19, made the first formulation of the equations of analytical dynamics according to the principles of the calculus of variations, and his approach was superior to Euler's semi-geometric methods. Thus, the classical Euler-Lagrange variational problem of determining the extremum value of a functional led to the celebrated Euler-Lagrange equation.

It has been calculated that his publications during his life averaged about 800 pages a year. His complete works entitled *Opera Omnia* consist of nearly 80 volumes, each approximately between 300 and 600 pages. Euler was undoubtedly the most prolific mathematical and physical scientists of all time. His whole working life was totally dedicated to the pursuit of fundamental discovery, dissemination of mathematical and scientific knowledge, and popularization of their value to common people. His famous three-volume *Letters to a German Princess on Different Subjects in Natural Philosophy* was one of the most popular books on science ever written and it was translated from German into eight different languages. The *Letters* addressed a wide variety of subjects including optics, acoustics, mechanics, astronomy, music, dioptrics, electricity and magnetism. This publication was essentially a unique encyclopedia of physical and philosophical ideas written in a popular style for the widest possible common audience. This work formed the basis for the reform of the teaching of physics and science. These are just a few examples of his prodigious contributions.

This volume is intended as a tricentennial memorial tribute to this universal mathematical scientist. My desire as well as interest in writing this book commemorating Euler's major contributions to mathematical and physical sciences is founded on the deep respect and admiration for him that I have gained from my own study and research of a small fragment of his voluminous work. The origin of this book was essentially based on my postgraduate course in the theory of elliptic functions and integrals with applications in 1960s. Indeed, I was further stimulated by my own articles and lectures for the last ten years on Euler and his major contributions. These publications and presentations are intended for the great majority of senior undergraduates and graduate students of mathematics, physics, and engineering.

The intense and narrow specialization of contemporary mathematics is a fairly recent phenomenon. The professional mathematical scientists spend almost all of their time and energy on segments of mathematics or science that seem to have little relationship to each other. They have hardly any time or opportunity to become familiar with the history of mathematics and science. The emphasis on history may provide more broad perspective on the whole subject and relate the subject matter of the courses not only to each other, but also to the major developments of mathematical thoughts. As Henri Poincaré eloquently wrote: "If you wish to foresee the future of mathematics, our proper course is to study the history and present condition of science." The writing of this volume was greatly influenced by the above thought of Poincaré. This book may serve to some extent as a historical introduction to mathematical sciences with the major emphasis on selected Euler's contributions. I hope that it will be helpful to professional and prospective mathematical scientists.

While writing this book as an exposition and survey of history, three major objectives have been kept in mind. The first is to focus each chapter on a subject to which Euler made a significant research contribution. Included are a short history of mathematical developments and discoveries before Euler, and a brief sketch of the life, work, career, and major achievements of Euler. The second is to present some historically significant, elegant, or unexpected theorems, proofs and results with applications. The third is to convey something of the fascination of mathematical sciences — of their beauty, intellectual power, and wide variety. This book does not require a graduate school mastery of any branch of mathematical sciences. It contains a wide variety of material accessible to the widest possible audience of mathematically literate readers.

It is my pleasure to express my grateful thanks to many friends, professional colleagues and students around the world who offered their suggestions and help at various stages of the preparation of the book. I am particularly grateful to my graduate students, Arunabha Biswas and Arindam Roy for helping me during the preparation of the book, especially for drawing all the figures in the book. My special thanks to Ms. Veronica Chavarria who cheerfully typed the manuscript with constant changes and revisions and carefully checked all the names in the text. In spite of the best efforts of everyone involved, some typographical errors will doubtlessly remain. I wish to express thanks to Ms. Lai Fun Kwong and the Production Department of Imperial College Press for their help and cooperation. Finally, I am deeply indebted to my wife, Sadhana, for her understanding and tolerance while the book was being written.

Lokenath Debnath
Edinburg, Texas

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Leonhard Euler (1707-1783): Chronology

April 15, 1707	Euler was born in Basel, Switzerland.
1720	At the age of 13, he graduated from the University of Basel with Philosophy Major.
1724	At the age of 17, he received his Master's degree with a thesis comparing the philosophy of René Descartes with that of Sir Isaac Newton.
1726-1727	Published first two research papers on the construction of isochronous curves in a resisting medium and on reciprocal algebraic trajectories.
1726-1741	Maintained a regular contact with Johann Bernoulli and his two sons Daniel and Nicholas.
1727-1741	Joined the Imperial Russian Academy of Sciences in St. Petersburg and worked with Daniel Bernoulli and Jacob Hermann. He was selected to become a Professor of Mathematics at the age of 26, and to be in charge of the Geography Department. This 14-year stay in St. Petersburg was the first golden period of his life.
1729	He first discovered the first fundamental function in real and complex analysis, known as the *gamma function* defined by the infinite integral

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \qquad \operatorname{Re} x > 0,$$

as a generalization of the factorial function.

He also introduced the Eulerian integral of the first kind, known as the *Euler beta function*, in the form

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \; x > 0, \; y > 0.$$

There is an elegant and beautiful relation between these two functions given by

$$B(x, y) = \frac{\Gamma(x)\Gamma(x)}{\Gamma(x+y)}.$$

1730 Euler discovered his celebrated zeta function for real s defined by an infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad s > 1.$$

The value of $\zeta(s)$ for $s = 1$ led him to discover the divergent harmonic series

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots \infty,$$

where each of its terms is the harmonic mean of the two neighboring terms.

1732 Euler stated memorable the *Euler-Maclaurin summation formula* which was independently discovered by Euler and Maclaurin. For a function $f(x)$ with continuous derivatives of all orders up to and including $(2m+2)$ in $0 \le x \le n$, then the sum $\sum_{k=0}^{n} f(k)$ is given by the Euler-Maclaurin summation formula

$$\sum_{k=0}^{n} f(k) = \int_0^n f(t)dt + \frac{1}{2}[f(0)+f(n)]$$

$$+\sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right] + R_m,$$

where B_{2k} are the Bernoulli numbers, and the remainder term R_m is

$$R_m = \tfrac{1}{(2m+1)!}\int_0^n B_{2m+1}(t) f^{(2m+1)}(t)dt.$$

1734 He married Catharina Gsell, daughter of a Swiss artist then working in Russia and they had 13 children and only 5 survived infancy.

1734-1737 Using the divergence of the harmonic series and the identity

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \prod_p \left(1 - \frac{1}{p}\right)^{-1},$$

Euler proved that the number of primes is infinite. Euler proved another remarkable theorem, for $s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where p is a prime. This establishes an unexpected link between the zeta function in analysis and the distribution of prime numbers in number theory.

1735 He discovered four distinct solutions of the Basel problem of finding the sum of the squares of the reciprocals of the integers, that is,

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}.$$

1736	Euler first solved the famous problem of the Seven Bridges of the city of Königsberg on the River Pregel that to determine a route around the city so that one can cross seven bridges once and only once. He proved that such a route is impossible. But with an extra bridge added, he proved that the solution is possible. This marked the beginning of a new area of mathematics known today as graph theory.
	He also published his two large volumes, *Mechanica sive motus scientia analytice exposita* (*Mechanics or the science of motion, expounded analytically*). The two-volume *Mechanica* dealt with a comprehensive treatment of almost all aspects of mechanics including the mechanics of rigid, flexible and elastic bodies as well as fluid mechanics, celestial mechanics and ballistics.
	He first discovered his celebrated equations which described the principles of conservation of mass, momentum, and energy. He then formulated the renowned Euler equations of motion for both incompressible and compressible inviscid fluid flows.
1738-1740	He won the Grand Prix of the Paris Academy and became an eminent mathematical scientist in the whole of Europe. He became blind in the right eye in 1738.
1738-1741	Political conditions of Russia became very unstable and the Russian Government was reluctant to support scientific research. He became concerned about his future in St. Petersburg. Euler left St. Petersburg in 1741 for the Berlin Academy in Germany.
1739	He published his treatise on the theory of music entitled *An attempt at a new theory of music, clearly expounded on the most reliable principle of harmony.*

1740 His other magnificent discovery was the *universal Euler constant* γ defined by the limit

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right)$$
$$= 0.577215665....$$

This constant was linked with the finite harmonic series and the logarithm function.

Euler single-handedly created the theory of partitions of numbers by a brilliant use of generating functions and formal power series. He surprised the mathematical community of the world with the remarkable expansion

$$\prod_{n=1}^{\infty}(1 - x^n) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{1}{2}(3n^2-n)}.$$

This led him to discover the *Euler Pentagonal Number Theorem* in number theory.

1741 At the invitation of the King Frederick the Great of Prussia, Euler joined the newly organized Berlin Academy of Science (originally founded by G. W. Leibniz in 1700).

1741-1766 Remained in Berlin Academy of Science for 25 years and completed his greatest work on Mechanics, Physics, Pure and Applied Mathematics. His 25-year stay in Berlin was regarded as the second golden period of his life.

1743 Two of the greatest discoveries are Euler's elegant and beautiful formulas,

$$e^{\pm ix} = \cos x \pm i \sin x,$$

and he then established two magnificent formulas

$$e^{i\pi} = -1 \quad \text{or} \quad e^{i\pi} + 1 = 0 \quad \text{and} \quad e^{2\pi i} - 1 = 0.$$

These simple formulas relate to six fundamental constants e, i, π, 0, 1 and -1 in mathematics and science.

1744 He was elected as Member of the Royal Society of London and to the Paris Academy of Sciences, among other many honors and awards.

He published his masterpiece treatise entitled *Methodus Inveniendi Lineas Curvas Maximi Minimive proprietate gaudentes sive solutio problematis isoperimatrici Latissimo Sensu Acceptl* (*A method for discovering curved lines that enjoy a maximum or minimum property, or the solution of the isoperimetric problem taken in its widest sense*) which contained his memorable extensive research in the theory of Calculus of Variations.

He published his major research monograph on *Theoria Motuum Planetarum et Cometarum* (*The Theory of Motion of Comets and Planets*) with solutions of major problems of theoretical astronomy with nature, structure, motion and action of comets and planets.

1745 He translated Benjamin Robins' 1742 treatise "*New Principles of Gunnery*" in German with a large extensive commentary appended to it. His revised and expanded version entitled *Artillerie* was published.

It was Euler who first made a serious attempt to study divergent series and integrals in a systematic manner. From the mathematical and physical point of view, Euler's ingenious work was very useful and served as the foundation of more modern theory of divergent series and integrals with physical applications.

1746 Euler's *New Tables for Calculating the Position of the Moon* was published in Berlin.

1748 He published his two-volume masterpiece treatise on mathematical analysis entitled *Introductio in Analysin Infinitorum* (*Introduction to the Analysis of the Infinite*).

1749 He completed the remarkable two-volume treatise *Scientia Navalis seu tractatus de construendis ac dirigendis navibus* (*Naval Science*) or *Ship building and Navigation* in Berlin.

Euler proved a beautiful formula for $\zeta(2n)$, where $n(\geq 1)$ is a natural number, and he also discovered a remarkable functional equation for the zeta function in the form

$$\zeta(1-s) = 2(2\pi)^{-s}\,\Gamma(s)\zeta(s)\cos\left(\frac{\pi s}{2}\right),$$

where $\Gamma(s)$ is the gamma function discovered by Euler in 1729.

1751 He established a major milestone through his extensive research and study of elliptic functions and elliptic integrals. He also proved many notable results which dealt with the addition and multiplication theorems of elliptic integrals. His brilliant work stimulated tremendous interest amongst many great mathematicians including Gauss, Lagrange, Jacobi, Abel, Galois, Weierstrass and Riemann.

1753 Euler published his Memoir on Ballistics with the first complete analysis of the equations of ballistic motion in the atmosphere. He also published his works on celestial mechanics and the fundamental monograph on the theory of lunar motion.

1755	He published his first comprehensive textbook on differential calculus entitled *Institutiones Calculi Differentialis* (*Foundation of Differential Calculus*).
1758	Euler discovered the celebrated formula $$V - E + F = 2 \quad \text{(the Euler characteristic)}$$ for a regular polyhedra and tried to prove it.
	He completed his notable masterpiece Memoir on the Calculus of Variations. In addition, Euler was involved in major administrative duties of the Academy including the Observatory, Botanic Gardens, Calendars and Maps.
1759-1766	He served as President of the Berlin Academy under the direct supervision of King Frederick the Great of Prussia who did not respect and trust him. The King Frederick offered the Presidency of the Academy to d'Alembert who was Euler's scientific rival. So, Euler became very concerned about his future career in Berlin.
1760	He first discovered the Euler phi-function $\phi(n)$ to generalize the Fermat Little Theorem in the form $a^{\phi(n)} \equiv 1 \pmod{n}$, where $(a, n) = 1$. This function has modern applications to a new area of mathematics known today as *cryptography* which deals with secure system transmission of secret messages and ciphers.
1760-1762	Euler wrote his famous *Letters to a German Princess, Anhalt-Dessau* on different subjects in natural philosophy, astronomy, optics, music, acoustics, electricity and magnetism that was one of the most popular science books ever written in the history of science.

1765	Euler published his great third volume of his mechanics book entitled *Theoria motus corporum solidorum seu rigidorum* (*Theory of Motion of Rigid Bodies*). It contained Euler's differential equations of motion of a rigid body under external forces. He introduced the original idea of employing two coordinates — one fixed, the other moving attached to the body, and first derived differential equations for the angles between the respective coordinates axes, now called the *Euler angles*. He worked out many major and interesting examples including the intriguing motion of the spinning of the top.
1766	At the age of 59, Euler received a cordial invitation from the German Princess, Catherine the Great of Russia, and moved back to St. Petersburg Academy.
1766-1783	His second St. Petersburg stay of 17 years can be regarded as the third golden period of his life. This period was very famous for his prolific and prodigious scientific activities as he completed a large number of epochal mathematical and scientific treatises and a highly successful and popular work on mathematics, science, and history and philosophy of science. It was also a time that Euler suffered from several major health problems and family disasters.
1768	Euler wrote his treatise on geometrical optics in three volumes and his tract on the motion on the Moon.
1768-1770	His three-volume textbook on integral calculus entitled *Institutiones Calculi Integralis* was published.
1769-1771	The three volumes of Euler's *Dioptrics* were published. This work dealt with his extensive research in optical sciences and optical instruments including telescopes and microscopes.

1770	Euler published his two-volume treatise on *Vollständige Anleitung zur Algebra* (*Elements of Algebra*).
1771	His house was badly burnt down in a fire. He lost his household, but most of his books and manuscripts were saved. He became almost blind due to an unsuccessful surgery to remove cataract in his left eye.
1773	He published his remarkable book *Théorie compléte de la construction et de la manoeuvre des vaisseaux* (*The Complete Theory of Ship Building and Navigation*) which contained the theory of the tides and the sailing of ships.
1773-1776	His wife, Catherina, died in 1773 after 40 years of their married life and he remarried to Catharina's half sister, Salmone Gsell, in 1776.
1776	Euler returned to mechanics with his seminal work on definite formulation of the principles of linear and angular momentum.
1776-1783	Euler completed almost half of his work during his most productive second 18-year stay at St. Petersburg. He continued his research on optics, algebra, geometry, celestial mechanics, naval science, lunar and planetary motion. In addition, he did some major research on probability theory and statistics, cartography, geography, chemistry, agriculture, pension funds, history of mathematics and science, medical and herbal remedies.
September 18, 1783	At the age of 76, Euler died in St. Petersburg as a result of a stroke while playing with his grandson.

Contents

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Chapter 1

Mathematics Before Leonhard Euler

> "Number rules the Universe."
>
> *The Pythagoreans*

> "Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line in extreme and mean ratio. The first we may name as a measure of gold, the second we may name as a precious jewel."
>
> *Johann Kepler*

> "As long as algebra and geometry proceed along separate paths, their advance was slow and their applications were limited. But when these sciences joined company, they drew from each other fresh vitality and hence forward marched on at a rapid pace towards perfection."
>
> *Joseph Louis Lagrange*

1.1 Introduction

Historically, mathematics originated from the fundamentals of counting in arithmetic. It is considered one of the greatest achievements of the human endeavor. Originally, it was the study of numbers or symbols and their relations. These symbols were created to stand for the *natural numbers* 1, 2, 3, $\cdots$ which form an infinite collection on which the basic arithmetic operations of addition and multiplication could be performed. It was the Ancient Hindus and Greeks who first discovered the natural numbers, but they did not acknowledge negative numbers. The first systematic algebra to use zero, negative numbers, and the decimal system was developed by

Hindu mathematicians in India during the seventh century A.D. They used positive and negative numbers to handle financial transactions involving credit and debit. Subsequently, mathematics has successfully been used to precisely formulate laws of nature.

Mathematics has more than 5000 years of history. By 3000 B.C., the people of Babylonia, China, Egypt and India had developed early and practical number systems. They used the knowledge of number systems in business, industry, government, science and indeed, in everyday life. Between 600 and 300 B.C., the Greeks took the next great step in advancing the knowledge of arithmetic, algebra, geometry and astronomy. They appear to have been the first to develop mathematical theory of arithmetic and geometry. Subsequently, it was realized that all scientific problems depend on mathematics for qualitative and quantitative descriptions and mathematical formulas became very useful for experiments and observations.

1.2 Pythagoras, the Pythagorean School and Euclid

Most of our knowledge of mathematics of the classical age came from the writings of many mathematicians and philosophers including Pythagoras (580-500 B.C.), Euclid (330-275 B.C.), Archimedes (287-212 B.C.) and Apollonius (260-200 B.C.). For the Greeks, mathematics was then largely synonymous with geometry which dealt with the measurement of land. Indeed, geometry was derived from two Greek words meaning *measurement of the earth*. The Ancient Egyptians used geometry to measure the size of their firm lands, and to find boundaries of these firm lands after yearly floods of the Nile River washed away or covered old landmarks. Classically, geometry dealt with the size, shape, area, volume or position of any object. More importantly, geometrical concepts and numerical ideas have been wrapped up together for thousands of years and they cannot be separated at all.

In about 540 B.C., Pythagoras established a school of mathematics and natural philosophy at Crotona in southern Italy. The influence of this great master Pythagoras was simply remarkable as his students and followers were very loyal to him and they formed themselves a society or brotherhood. They were known as the Order of the Pythagoreans. Members of the Pythagorean School were very obedient and loyal to their great master, shared everything in common, held the same religious and philosophical beliefs, made a commitment to the same pursuits and bound themselves to an oath not to reveal their own secrets and teachings of the school.

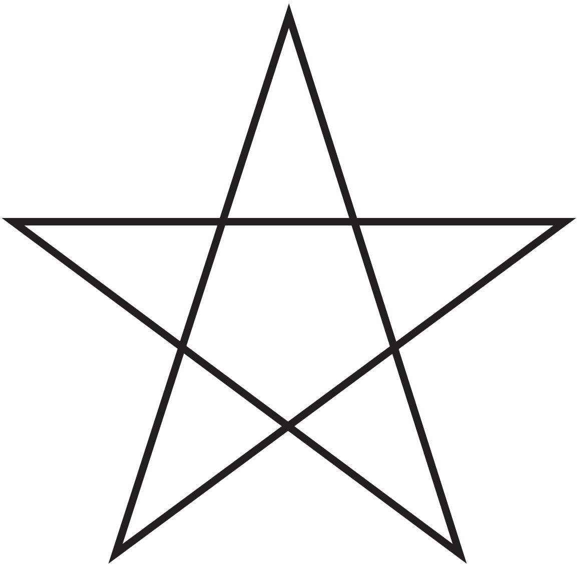


Fig. 1.1 The star pentagon.

It is remarkable that the school discovered a beautiful *star pentagon* or *pentagram* (see Figure 1.1), the most fitting badge of the Pythagorean brotherhood. It was also a fitting symbol of mathematics and the Greek emblem of health. In addition to their unique contributions to mathematics, particularly, to geometry and number theory; the Pythagoreans were specially interested in the study of medicine and music. Figure 1.2 shows an infinite sequence of nested pentagons.

They developed a large body of knowledge in geometry and properties of numbers, and proved a large number of geometrical theorems including one of the most famous theorems in geometry known as the *Pythagorean Theorem*:

$$c^2 = a^2 + b^2 \tag{1.2.1}$$

for any right-angled triangle of sides a and b adjoining the right angle and c is the hypotenuse.

This theorem has probably received more diverse proofs than any other theorem in all of mathematics. In the second edition of his book entitled The Pyathgorean Proposition, E. S. Loomis (1968) has reported about 367 demonstrations (or proofs) of this famous theorem. Making reference to Figure 1.3, a dissection type proof of this famous theorem can be given as

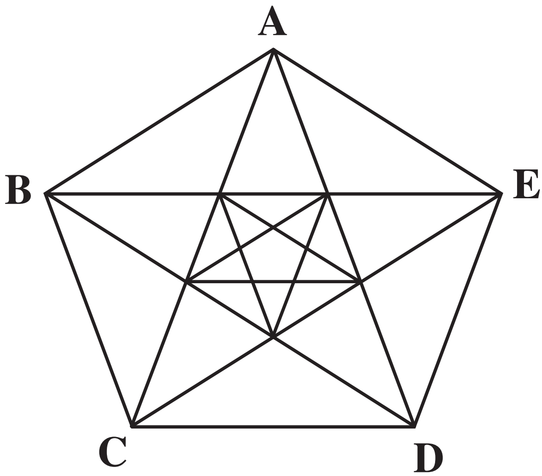


Fig. 1.2 Sequence of nested pentagons.

follows. The first square of side $(a+b)$ is dissected into four equal right angled triangles of sides a and b and a square of side c so that $(a+b)^2 = 4(\frac{1}{2}ab) + c^2 = 2ab + c^2$. The second figure is dissected into two squares and four equal right angled triangles so that $(a+b)^2 = 4(\frac{1}{2}ab) + a^2 + b^2$. Equating two equal expressions readily gives (1.2.1).

One of the Indian mathematicians, Bhaskara gave a second proof of the Pythagorean theorem by drawing the altitude on the hypotenuse of the

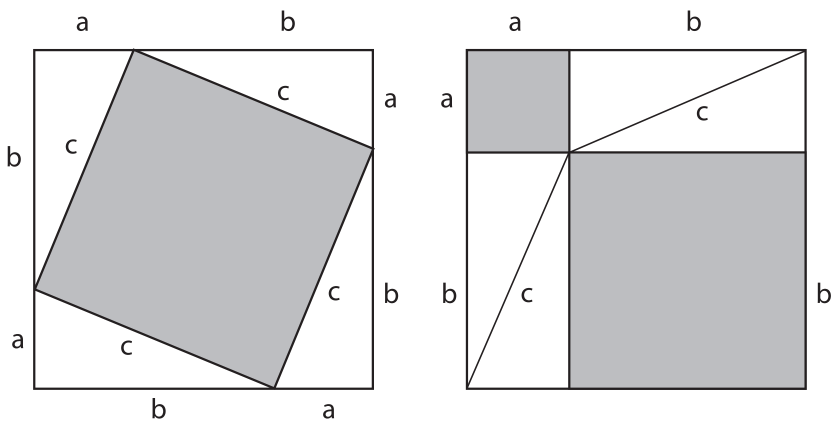


Fig. 1.3 Dissection of two equal squares.

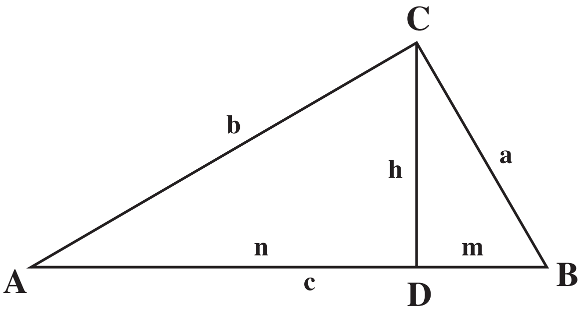


Fig. 1.4 A right angled triangle with $\angle ACB = 90°$.

right angled triangle ABC with $\angle ACB = 90°$. It follows from similar right angled triangles as shown in Figure 1.4 that

$$\frac{a}{c} = \frac{m}{a} \quad \text{and} \quad \frac{b}{c} = \frac{n}{b}.$$

Bhaskara gave another proof using dissection in which the square on the hypotenuse is divided into four equal triangles, (see Figure 1.5) each is congruent to the given right angled triangle of sides a, b, and c and a square with side $b - a$. Clearly, a simple algebra supplies the proof as follows:

$$c^2 = 4\left(\frac{1}{2}ab\right) + (b-a)^2 = a^2 + b^2$$

or

$$a^2 = cm \quad \text{and} \quad b^2 = cn$$

so that

$$a^2 + b^2 = c(m+n) = c^2.$$

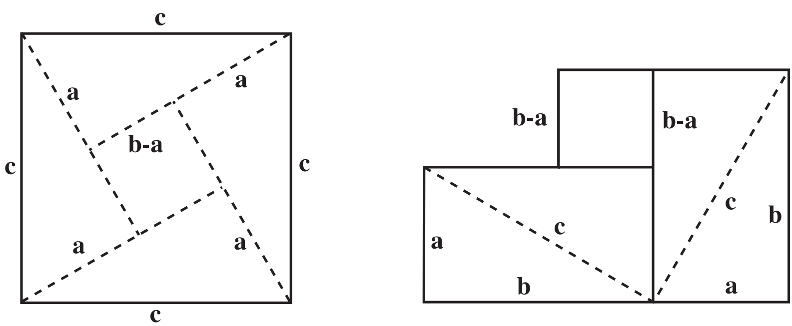


Fig. 1.5 Dissection of a square into four triangles and a square.

A famous British mathematician, John Wallis (1616-1703) rediscovered this ancient proof in the seventeenth century. For several other proofs of the Pythagorean theorem, the reader is referred to Loomis' book (1940) and to a book entitled *Great Moments in Mathematics Before 1650* by Howard Eves (1911-2004) published in 1983.

When $a = b = 1$, $c = \sqrt{2}$, an *irrational number* which cannot be expressed as the ratio of two integers. In other words, the rational numbers are not adequate for measuring the hypothenuse of a right-angled triangle whose base and height are unity. The discoveries of Pythagoras theorem and the irrational numbers were the greatest achievements of the Pythagoreans in the history of mathematics. Indeed, the Pythagorean discovery of an irrational number led to the solution of equations such as

$$x^2 = 2. \tag{1.2.2}$$

Although $x = \sqrt{2}$ is irrational, but it can be expressed in terms of approximate rational numbers 1.4, 1.41, 1.414, $\cdots$ with finite number of decimal places.

The Pythagoreans also proved many geometrical theorems including the equality of the base angles of an isosceles triangle, and the sum of three angles of a triangle is equal to two right angle. They also proved the famous algebraic identities

$$(a \pm b)^2 = a^2 \pm 2ab + b^2 \tag{1.2.3ab}$$

using purely geometrical arguments.

More remarkably, they made three great discoveries: the first, one was the introduction of *proof* in mathematics, that mathematical proof must proceed from given assumptions, the second one was that the natural numbers were insufficient for the construction of mathematics, and the third one was that the set of natural, rational and irrational numbers form the complete set of real numbers with the geometrical interpretation. Geometrically, to each real number corresponds to one and only one point on the real line. In addition, there were three famous unsolved problems that exerted so great influence on the development of Greek mathematics. The original idea was to solve them by *ruler and compasses* constructions. However, the impossibility of solutions by a ruler and a compass kept these problems at the center of the mathematical stage for many centuries.

The first problem was known as the *Delian problem* which dealt with the doubling of a cube, that is, to construct a cube whose volume is twice that of a given cube. Mathematically, the problem is to find a solution of

$$x^3 = 2. \tag{1.2.4}$$

In about 400 B.C., Archytas brilliantly solved the problem by finding the point of intersection of three surfaces in three-dimensional space: a cylinder, a cone, and a torus generated by rotating a circle about one of its tangents. This was indeed a most remarkable achievement of Archytas as there was little known then about three-dimensional (or solid) geometry.

The second problem was the trisection of a given angle θ by a ruler and a compass. Mathematically, it reduces to a solution for θ which satisfies the equation

$$4x^3 - 3x - \cos\theta = 0. \tag{1.2.5}$$

For $\theta = 60^\circ$ so that $\cos\theta = \frac{1}{2}$. The polynomial on the left of (1.2.5) is irreducible over the field Q of rational numbers. It can be shown that θ cannot belong to a field extension E of Q of degree 2^m. Consequently, the trisection of an angle $\theta = 60^\circ$ is *not* possible with a ruler and a compass. For the construction of regular polygons with a ruler and a compass, the set of complex solution of the well-known *cyclotomic equation*

$$x^n - 1 = 0 \tag{1.2.6}$$

contains the number one and divides the unit circle into n equal parts. The solution is possible with the aid of the following theorem due to Gauss:

Theorem of Gauss: A regular n-gon can be constructed with ruler and compass if and only if

$$n = 2^m p_1 p_2 \cdots p_r, \tag{1.2.7}$$

where m is a natural number and $p_r's$ are pair distinct Fermat's primes of the form

$$F_k = 2^{2^k} + 1, \qquad k = 0, 1, 2, \cdots. \tag{1.2.8}$$

It is probably known that for $k = 0$, 1, 2, 3, 4, the above number is prime. Consequently, a regular n-gon can be constructed for n in the list of primes 2, 3, 5, 17, 257, 65, 537. For $n \leq 20$, the construction of all regular n-gons with $n =$ 3, 4, 5, 6, 8, 10, 12, 15, 16, 17,20 is possible using only a ruler and a compass.

Finally, the third problem was to square the circle, that is, to construct a square with ruler and compass whose area is equal to that of a given unit circle. The length x for the sides of a square is a solution of the equation

$$x^2 = \pi. \tag{1.2.9}$$

In 1882, Ferdinand Lindemann (1852-1939), David Hilbert's (1862-1943) teacher, proved the transcendence of the number π over the field Q of rational numbers. Consequently, number x or π cannot be an element of any algebraic field extension of Q. So the problem has no solution. Clearly, the first two problems are algebraic so that they require the solution of a cubic equation. The third problem is totally different as it involves the transcendental number π.

Indeed, in mathematics, the Pythagoreans made great progress, particularly in the theory of numbers, and in geometry of line, plane and solid figures, and also, lengths, areas, and volumes associated with them. It is the most appropriate to recall the delightful quotation of Johann Kepler (1571-1630): "Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line in extreme and mean ratio. The first we may name as a measure of gold, the second we may name as a precious jewel."

In Greek mathematics, there was another remarkable number, the so called the *golden number* (or *golden ratio*) that is defined in geometry by dividing a straight line segment in such a way that the ratio of the total length l to the larger segment x is equal to the ratio of the larger to the smaller segment. In other words, the golden ratio, $g = (l/x)$ is determined by the equation

$$\frac{l}{x} = \frac{x}{l-x} \tag{1.2.10}$$

or, equivalently,

$$g^2 - g - 1 = 0. \tag{1.2.11}$$

The positive solution of quadratic equation (1.2.11)

$$g = \frac{l}{x} = \frac{1}{2}\left(\sqrt{5}+1\right) = 1.618. \tag{1.2.12}$$

The inverse ratio of g is

$$\frac{1}{g} = \frac{x}{l} = \frac{1}{2}\left(\sqrt{5}-1\right) = 0.618 \tag{1.2.13}$$

so that $\frac{1}{g} = g - 1$.

In geometry, the Pythagoreans developed the theory of space filling figures, whatever the motivation for their work, the Pythagoreans evidently considered the geometrical figures to be very important for space filling. For example, one of the diagrams (see Figure 1.6) shows six equal equilateral triangles filling space around their central point. But five such equilateral triangles can similarly be fitted together to generate a bell-tent-shaped figure around a central vertex so that their bases form a regular pentagon.

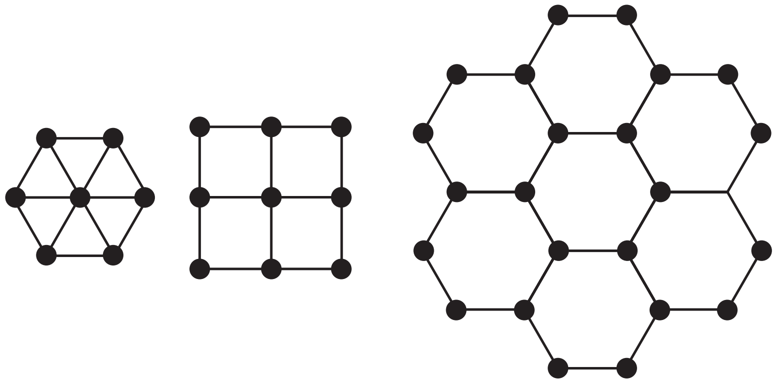


Fig. 1.6 Pythagorean or Six Equilateral Triangles filling space around the center.

Such a figure becomes a solid figure with the vertex of a regular *icosahedron*. This process can be repeated by surrounding each vertex of the original equilateral triangles with five triangles. Exactly twenty equilateral triangles are required to generate the beautiful solid figure of the icosahedron of twelve vertices and twenty faces. It is remarkable that in solid geometry there are exactly five such regular figures, known as *regular polyhedra* (or *Platonic solids*), and that in the plane there is a very limited number of regular space-filling geometric figures. The first three simplest regular polyhedra including *tetrahedron*, *cube* and *octahedron* were found by Egyptian mathematicians. Pythagorean discovered the remaining two — the icosahedron, and the *dodecahedron* with twenty vertices and twelve faces.

It is important to point out that a study of the properties of the regular pentagon led to the discovery of the golden ratio, the ratio in this case being that of the diagonal of the pentagon to its side. In Figure 1.7, the diagonal AC of the pentagon divides the diagonal BE into two unequal segments BP and PE such that the ratio of the smaller segment to the larger is equal to the ratio of the larger segment to the whole diagonal. In fact, any diagonal of the regular pentagon divides any other interesting diagonal in this way. Such division was known to the Greek mathematicians as "division of a line in mean and extreme ratio". We have already stated that this ratio is the golden ratio $g = (BE/PE)$, where $BE = \ell$ and $PE = x$ so that the algebraic formulation is $(\ell - x)/x = (x/\ell)$. This leads to the quadratic equation (1.2.11) in the golden ratio g.

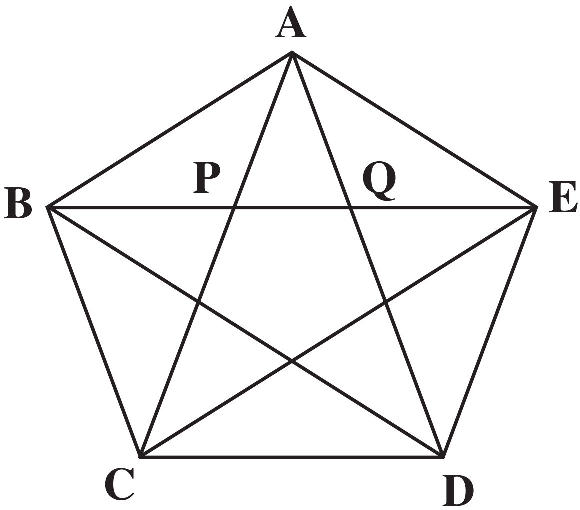


Fig. 1.7 A Regular Pentagon $ABCDE$.

Some of the angles associated with Figure 1.7 follow from the construction of the triangle ACD with angles $\angle ACD = \angle ADC = 72°$ and $\angle CAD = 36°$. It is then a simple matter to construct the complete pentagon so that $\angle ABC = 108°$, $\angle BAC = 36° = \angle BCA$, and hence, all angles of the pentagon are known.

It was Euclid of Alexandria (365-300 B.C.) made the first systematic development of Euclidean geometry in his famous treatise, *The Elements* in 13 volumes. These volumes represented a standard reference of geometry and number theory and a great model for the first axiomatic method in mathematics. He first proved that the number of primes is infinite which is one of the fundamental results in mathematics. However, the first completely rigorous axiomatic method of mathematics from a modern point of view was given by David Hilbert in his *Principle of Geometry* that was published in 1899. The Greek mathematicians also advanced other areas of mathematics and astronomy. Archimedes (287-212 B.C.) of Syracuse also made many other major contributions to mathematics and mathematical physics. He determined the center of mass of bodies and simple surfaces and derived the formula for the workings of levers and equilibrium of floating bodies. His major work for finding areas and volumes marked the birth of calculus. Archimedes was probably the last great mathematical scientists of ancient times. Another Greek mathematician, Claudius Ptolemy (85-169 A.D.) of Alexandria became famous for his major contributions to plane and spherical trigonometry and astronomy. Diophantus of Alexandria worked on theory of equations and earned the title of Father of Algebra. During the Middle Ages, the greatest discoveries in India were natural numbers including zero and the decimal number system. According to P. S. Laplace (1749-1827): "It is India that gave us the ingenious method of expressing all numbers by ten symbols, each symbol receiving a value of position, as well as an absolute value. We shall appreciate the grandeur of the achievement when we remember that it escaped the genius of Archimedes and Apollonius."

1.3 The Major Impact of the European Renaissance on Mathematics and Science

During the middle ages, the Italian mathematician Leonardo of Pisa (Fibonacci (1170-1250)) published his major book *Liber Abaci* in 1202, an influential book which introduced the Hindu-Arabic number system to Western Europe. The *European Renaissance*, from the 1400 to the 1600s produced many great advances in physics, astronomy, pure and applied mathematics. Michael Stifel (1487-1567), Nicolò Tartaglia (1506-1557), Girolamo Cardano (1501-1576), and Francois Viète (1540-1603) made major contributions to algebra, trigonometry and quadratic and cubic equations. Viète introduced the use of letters to stand for unknown quantities. Nicolaus Copernicus (1473-1543), the great astronomer who boldly rejected the fourteen-hundred year old Ptolemy's mathematical theory of astronomy with the Earth at the center of the universe and discovered the revolutionary modern heliocentric picture of the universe with the Sun at the center and made contributions to mathematics through his great work in astronomy with the publishing of *De revolutionibus orbium coelestium* in 1543.

Thoroughly convinced by the beauty and harmony of the Copernicus heliocentric system that the planets revolve in orbits about the sun at the center of the Universe, a great German mathematical scientist and astronomer, Johann Kepler used his brilliant imagination and amazing perseverance to modernize the Copernicus model in mathematical astronomy. As a research assistant to the famous Danish-Swedish astronomer, Tycho Brahe (1546-1601), Kepler had the rare opportunity to utilize Brahe's precise and extensive observational data. Based on these observational data, Kepler first discovered his three famous laws of planetary motion, the first two founded in 1609 and the third one ten years later in 1619. Kepler's laws of planetary motion are considered as major landmarks in the history of mathematical science and astronomy, for in the effort to justify them, Newton was led to discover modern celestial mechanics during 1660-1666. In his celebrated work of 1619, *Harmony of the World*, Kepler expressed his great satisfaction with the following statement in the preface:

"I am writing a book for my contemporaries or — it does not matter — for posterity. It may be that my book will wait for a hundred years for a reader. Has not God united for 6000 years for an observer?"

The 1600s brought many major discoveries in mathematics and astronomy. Two British mathematicians, John Napier (1550-1617) and Henry Briggs (1556-1631), first Savilian Professor of Geometry at the University of Oxford, invented logarithms to the base of 10. Logarithms to the base of 10 are usually known as Briggian logarithms, through the advantage of using this base appears to have occurred independently to Napier and Briggs. Napier published his book *Mirifici logarithmorum canonis descriptions*, in which logarithms are introduced in great detail. On the other hand, two Englishmen, Thomas Harriot (1560-1621), and William Oughtred (1557-1660) developed new methods for classical algebra. Galileo Galilei (1564-1642) an Italian astronomer and physicist and Johann Kepler, a German mathematician and astronomer tremendously expanded knowledge of mathematics and physics through their studies of astronomy, physics and mathematics. Galileo discovered the famous law of falling bodies which marked the beginning of modern experimental physics. He suggested that all bodies are attracted to the Earth by the constant gravitational acceleration regardless of their weights. His famous experiment dealt with dropping two unequal weights from the top of the Leaning Tower of Pisa. This became controversial because it contradicted Aristotle's (384-322 B.C.) old views that heavy bodies fell faster than lighter ones. It is also important to mention Galileo's work on the curve *cycloid* in 1630 and his suggestion that arches of bridges should be built in the shape of cycloid. The quadrature (or finding an area) of the cycloid has been calculated in 1630 by Evangelista Torricelli (1608-1647), a student of Galileo. About the same time, Pascal proved many new theorems about properties of cycloid and calculated the area of the segment of cycloid. This was followed by another remarkable discovery of a great Dutch mathematical scientist, Christian Huygens (1629-1695) in 1658 that was concerned with the solution of the problem of the tantochronous motion. Indeed, the cycloid is a true tantochrone that is, if a particle is allowed to slide from rest down a cycloid, it takes exactly the same time to reach the bottom, no matter where it starts from. Huygens also made another discovery that a pendulum bob swinging along a cycloid curve takes exactly the same time to make a complete oscillation whether it swings through a small or large arc. He made many new and sensational discoveries in physics and astronomy including his strong support for the Copernicus heliocentric model of the universe. In 1609, he build a telescope that has opened new worlds in astronomy, and has become an indispensable instrument for centuries for astronomy.

Galileo discovered the laws of pendulum and was credited for his most remarkable discovery of four bright satellites of the planet Jupiter in 1610. In the same year, he observed some peculiar form of the planet Saturn. His historic achievements in astronomy dealt with the discovery of many more and more powerful telescopes that were sold in Europe. This instrument has made it possible to study, observe and photograph many heavenly bodies which were formerly unknown. His name and fame as the greatest experimental scientist of his time attracted many scholars from all parts of Europe. Christian Huygens also built a powerful telescope which made possible his new discovery of satellites and the rings of Saturn. He was the first one who used a pendulum to regulate a clock and then applied the basic principles of pendulum in building astronomical clocks. In addition, he investigated the wave theory of light and discovered the polarization of light.

Galileo is universally considered as the founder of methodology of modern science. His radical departure from the Greeks, the medieval and contemporary scientists led him to establish the fact that matter as well as motion were only the first step to a new approach to nature. In 1632, he published his beautifully written masterpiece, *A Dialogue on the Two Principal Systems of the World* in which he gave a critical evaluation of the comparative merits of the old and new theories of motions of the celestial bodies. He spent considerable amount of time in writing on force and motion. In particular, his firm helicentric views of the universe was in severe disagreement with religion doctrines of the Inquisition. In 1638, he published his other greatest classic, *Dialogues on the Two New Sciences* in which he founded the modern science of mechanics. It contained his life's work on motion, acceleration, and gravity and provided a sound basis for the three laws of motion formulated by Sir Isaac Newton (1642-1727) in 1687.

Without being too precise, mechanics is simply the study motion of material bodies (or particles) that can be described by mathematical models. In mechanics, a body (particle) is supposed to be subject to certain forces, which affect its motion according to certain laws. Expressed in the language of mathematics, these laws usually take the form of differential equations, that is, they connect the position, velocity, momentum, acceleration of the body at a particular instant of time. They do not primarily describe the whole motion, but merely the laws governing it. It is the motion as a whole which has to be derived from the law. In other words, this is a problem of solving differential equations with time as the independent variable, and there are one or more dependent variables which determine the position of the body.

Galileo's two greatest classics are not only two profound books of all time, but they are clear, direct, truly powerful and fascinating in the history of science and philosophy. In general, his scientific philosophy and scientific method were in agreement with those of Descartes, Huygens, Newton and others. His new methodology of science led him to believe in the total reformulation that not only imparted expected and unprecedented power to science, but bound it indissolubly to mathematics. It was Galileo who remarkably discovered the more radical, more effective and more practical methods for modern science. He demonstrated the profound effectiveness of his approach to science through his own work. It is a delight to quote a philosopher, Thomas Hobbes (1588-1678) who said of Galileo: "He has been the first to open to us the door to the whole physics." Galileo himself was convinced that nature is simple, orderly, and mathematically designed which can be documented by his own famous 1610 quotation: "Philosophy [nature] is written in that great book which ever lies before our eyes — I mean the universe — but we cannot understand it if we do not first ... labyrinth."

Both Galileo and Newton strongly emphasized that mathematical principles are quantitative principles which played a vital role in providing the correct physical explanation of natural phenomena. They also believed that experiments are needed to establish basic laws of science. In the preface to his *Principia*, Newton expressed his firm views on the intimate relationship between the mathematical principles (or laws) and the natural phenomena as follows:

"Since the ancients (as we are told by Pappus) esteemed the science of mechanics of greatest importance in the investigation of natural things, and the moderns, rejecting substantial forms and occult qualities, have endeavored to subject the phenomena of nature to the laws of mathematics, I have in this treatise cultivated mathematics as far as it relates to philosophy [science] ... and therefore I offer this work as the mathematical principles of philosophy, for the whole burden in philosophy seems to consist in this — from the phenomena of motions to investigate the forces of nature, and then from these forces to demonstrate the other phenomena....”

Finally, we close this section by adding the most tragic event of Galileo’s life. In 1633, after a long and painful trial by a tribunal of the Inquisition because of his heliocentric view of the universe contrary to church teachings, he was sentenced to house imprisonment for the rest of his life. He remained a prisoner in Florence until his death in 1642.

During the Renaissance period, two great French mathematicians, René Descartes (1596-1650) and Pierre de Fermat (1601-1665) created a new branch of mathematics which is now known as *analytic geometry*. By the 1630s, both men discovered the basic idea of using algebraic equations to represent curves and surfaces and investigated their fundamental properties. Descartes’ major objective was to unify the hitherto largely two separate disciplines of algebra and geometry, in particular to use the algebraic method to solve geometrical construction problems. His great mathematical work dealing with applications of algebra to geometry was *La Géométrie*.

On the other hand, based on the work of Apollonius on conic sections, Fermat discovered the fundamental principle of geometry, which he expressed thus: “Whenever in a final equation two unknown quantities are found, we have a locus, the extremity of one of these describing a line, straight or curved.” This profound statement was written at least one year before the publication of Descartes’ *La Géométrie*. Fermat formulated his major ideas further in his short book entitled *Ad locus planos et solidos isagoge* (*Introduction to Loci Consisting of Straight Lines and Curves of the Second Degree*) which was published in 1679 – almost fourteen years after his death: That is why Descartes is widely recognized as the sole creator of coordinate geometry. However, it clearly follows from Fermat’s above quotation that his approach is undoubtedly more simple, direct and more systematic than that of Descartes. In the eighteenth century, the view that the algebraic approach to geometry was more than a mathematical tool. Algebra itself is a fundamental method of introducing and investigating curves and surfaces in general. All these simply mean that the analytic geometry paved the way for a complete unification of algebra and geometry from a modern point of view.

Based on the great work of the classical masters, Apollonius and Diophantus on geometry, in general and conic sections, in particular, Girard

Desargues (1591-1661) created a totally new branch of geometry in 1639 which is now know as the *Projective Geometry*. It deals with the study of the descriptive properties of geometrical figures. In other words, it is basically concerned with those geometrical properties which are unchanged by the operation of section and projection. The basic metrical properties in geometry which include distance, areas, angles, congruence, and similarity are not considered in projective geometry. For example, the Pythagorean Theorem for a right angled triangle ($c^2 = a^2 + b^2$), the area of a triangle of base a and height h $\left(\triangle = \frac{1}{2}ah\right)$ and the sum of the three angles of a triangle ABC $(A + B + C = \pi)$ are famous metrical theorems. Projective geometry has grown into a vast and beautiful branch of geometry through the major works of great French mathematicians including Desargues, Blaise Pascal (1623-1662), Gaspard Monge (1746-1818) and Jean Victor Poncelet (1788-1867). Like several other branches of geometry, it has become a new source of mathematical knowledge for the study of descriptive geometry.

The observed symmetry between points and lines in a projective plane leads to the so-called *principle of duality* which is one of the most elegant properties of the projective geometry. This basic principle states that, in a projective plane, every theorem (or proposition) remains true when the words *point* and *line* are consistently interchanged. Thus, given a theorem and its proof, we *can* immediately formulate the dual theorem whose proof can be written down mechanically by the use of the duality principle in every step of the proof of the original theorem.

Desargues not only introduced many ideas, notably the point and the line at infinity and gave elegant proofs of many new theorems. Above all, he first discovered the concepts of section, projection and cross-ratio of four points which were used to give a new method of proof. He then developed a unified approach to several types of conic sections through projections and sections. It may be appropriate to give some examples of basic theorems in projective geometry. One such example is the Desargues' famous two-triangle theorem which is illustrated in Figure 1.8 with a vortex O and the triangle $A^{'}B^{'}C^{'}$ is obtained from the triangle ABC by projection and section from the vertex O. Desargues' theorem then states that the three pairs of the corresponding sides AB and $A^{'}B^{'}$, BC and $B^{'}C^{'}$, and AC and $A^{'}C^{'}$ (or their extensions) of two triangles perspective from a point meet in three colinear points L, M, N as shown in Figure 1.8. Conversely, if the three pairs of corresponding sides of the two triangles meet in three points that lie on one straight line, then the line joining corresponding vertices meet in one point. In other words, making reference to Figure 1.8,

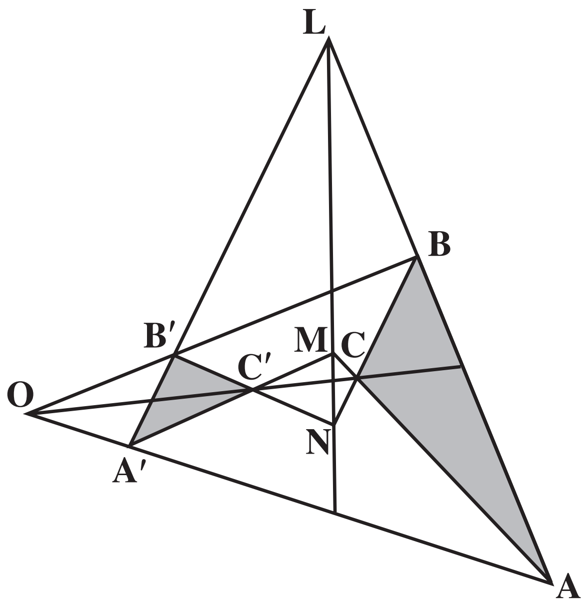


Fig. 1.8 The Desargues two triangles.

the converse theorem asserts that since $AA^{'}$, $BB^{'}$, and $CC^{'}$ intersect at a point O, the sides AB and $A^{'}B^{'}$ intersect at a point L, AC and $A^{'}C^{'}$ meet in a point M and BC and $B^{'}C^{'}$ meet at N; then L, M, and N lie on a straight line. It is important to note that both theorems are true whether the triangles ABC and $A^{'}B^{'}C^{'}$ lie on the same or different planes. Desargues gave an elegant proof of his theorem and its converse for both two- and three-dimensional cases.

The second major contributor to projective geometry was Pascal. In 1640, at the age of sixteen, Pascal gave a pleasant surprise to the world by publishing a short book entitled *Essai pour les coniques* in which he described his celebrated theorem of the *hexagrammum mysticum (Mystic Hexagram)*. It is universely known as the *Pascal Theorem* which is illustrated in Figure 1.9, and it states that the three pairs of opposite sides of a hexagon inscribed in a conic meet in three collinear points. In other words, making reference to Figure 1.9, if BA and DE intersect at L, CD and AF intersect at M, and BC and FE intersect at N, then L, M, N lie a straight line. Conversely, if a hexagon is such that the points of intersection of its three pairs of opposite sides lie on a straight line, then the vertices of the

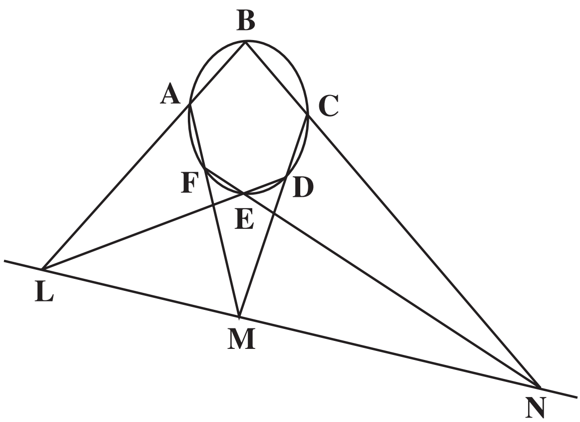


Fig. 1.9 The Pascal hexagon in a conic.

hexagon lie on a conic. However, Pascal did not give an explicit proof of his theorem and its converse. He simply stated that his theorem is true for a circle and it must also be true for a conic by the method of projection.

Desargues regarded Pascal's 1640 essay was so brilliant that he could not believe it was written by such a young man. He encouraged Pascal to do more research on projective geometry in order to develop the method of projection and section further. At the advice of Desargues, Pascal began working on conics and used projective methods, that is, projection and section. He admired Desargues' work and acknowledged his debt to Desargues by saying : "I should like to say that I owe the little that I have found on this subject to his writings."

In addition to his contribution to projective geometry, Pascal made major contributions to the mathematical theory of probability. In 1654, a French man, the Chevalier de Méré, suggested some problems associated with games of chance. During that time, Pascal had some correspondence with Pierre de Fermat dealing with these problems of games of chance and gambling in general. Thus, the first research collaboration of Pascal and Fermat on problems of games and chance led to mark the birth of the mathematical theory of probability which is now widely used in mathematical statistics. Based on some results of Fermat and Pascal, Christian Huygens wrote the first treatise on probability in 1657. About the same time, Pascal wrote a treatise entitled *Traité du triangle arithmétique* which included the coefficients of the binomial expansion

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} \ a^{n-r} \ b^r, \tag{1.3.1}$$

so that, when $a = b = 1$,

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n, \tag{1.3.2}$$

the number of *combinations* of n objects taken r at a time is,

$${}^nC_r = \binom{n}{r} = \frac{n!}{(n-r)!\ r!}, \tag{1.3.3}$$

the Pascal recurrence relation

$${}^nC_r +^n C_{r-1} = \ {}^{n+1}C_r, \tag{1.3.4}$$

and the coefficients of the binomial formulas organized into famous *Pascal's triangle*. Pascal made an extensive study of the properties of his triangle, in the course of which he discovered the *principle of mathematical induction*. This principle, which states the validity of the mathematical argument by recurrence, is now considered as one of the fundamental axioms of modern mathematics. Many proofs in mathematics are based on the famous *principle of induction*. Pascal became a renowned scientist in Europe for his fundamental works in geometry, hydrostatistics and probability theory. He also invented a new calculating machine which is still preserved in a French museum.

After a century of slow progress, the revival of the projective geometry received considerable attention by Gaspard Monge (1746-1818) and his school at the École Polytechnique. Monge's extensive work in descriptive geometry, ordinary and partial differential equations won the remarkable admiration from mathematical scientists of the world. His greatest student was Poncelet who published his famous *Treatise on the Projective Properties of Figures* in 1822 which he subsequently expanded and revised this treatise and later published in two volumes entitled *Applications d'analyse et de géométrie* (1862-1864). All these published works were his major contributions to projective geometry and to the creation of modern projective geometry. He was the first mathematician to recognize fully that projective geometry was a new branch of geometry with methods and results of its own. He formulated the general problem of seeking all properties of geometrical figures which were common to all sections of any projection of a figure, that is, remained unchanged by projection and section. His work was essentially based on three major ideas: homologous figure, principle of continuity, and pole and polar with respect to a conic. Two figures are called *homologous* if one can be derived from the other by one projection and a section. In his 1822 *Traité*, Poncelet phrased the principle of continuity as: "If one figure is derived from another by a continuous change and the latter is as general as the former, then any property of the first figure can be asserted at once for the second figure." He advanced the principle as an absolute truth and used it in his *Traité* to prove many theorems, and then generalized the principle to make assertions about imaginary figures. The concept of pole and polar with respect to a conic was the third major idea in Poncelet's work. He gave a general formulation of the transformation form pole to polar and conversely. His major objective in studying polar reciprocation with respect to a conic was to establish the principle of duality in projective geometry. By virtue of this principle, lines can be as fundamental as points in the development of plane projective geometry. Like others, Poncelet recognized that theorems dealing with figures lying in one plane when interchanged the word point by line and vice versa not only made sense but proved to be true in general. It is fair to say that all major contributors to projective geometry made the significant efforts to elevate the subject to hew heights of rigor, clarity, elegance and generality.

The Renaissance mathematical scientists including Copernicus, Brahe, Kepler, Galileo, Pascal, Huygens, Descartes, Newton and Leibniz spoke repeatedly of the cohesiveness and harmony that God imparted to the Universe through His *mathematical laws* and *design*. These men discovered mathematical knowledge that would reveal the grandeur and glory of God's creation. Once Galileo said: "Nor does God less admirably discover Himself to us in Nature's action than in the Scriptures' sacred dictions." Towards the end of the Renaissance period, many European scientists became very active in the formation of scientific societies or research institutes in order to stimulate more scientific research and to increase communication among mathematical scientists. Although the Italian academies and professional societies were founded in the early seventeenth century with Galileo and his students as members, but, unfortunately, they were disbanded after a while. For example, in France, several mathematical scientists including Desargues, Descartes, Fermat and Pascal met informally under the leadership of Marin Mersenne (1588-1648) to organize the Academie Royale des Sciences in 1630s. In England, John Wallis began in 1645 to hold meetings in Greshan College, London in order to establish a similar organization in England. This informal group was chartered by Charles II in 1662 and established the Royal Society of London for the promotion and dissemination of scientific knowledge. Its first president was a famous mathematician, Lord William Brouncker (1620-1684). *The Philosophical Transactions of the Royal Society* began its publication in 1665 and it was one of the first research journals to include mathematical and scientific papers. The French Academy of Sciences was founded by Colbert in 1666. The famous Lucasian Chair of Mathematics was established at the University of Cambridge in 1663 by Henry Lucas (1610-1667) who was a former student of Cambridge. The first professorship in mathematics was established at the University of Oxford in 1619. John Wallis became of professor of mathematics at Oxford in 1649 and held the Chair of mathematics until 1702. On the other hand, Gottfried Wihelm Leibniz (1646-1716) in Germany provided a major leadership role for some years to establish the Berlin Academy of Sciences in 1700 with Leibniz as its first President. In Russia, Peter the Great founded the Academy of Sciences at St. Petersburg in 1724. These academies and their scientific journals opened new outlets for mathematical and scientific communication first in Europe and then in other nations of the world. They not only promoted new scientific research, but also supported scientists for the cultivation of mathematics and sciences and for making mathematics and science more useful for the society. These professional organizations played the major role in advanced study and research, and in dissemination of scientific and mathematical knowledge throughout the world.

1.4 The Discovery of Calculus by Newton and Leibniz

Historically, Sir Isaac Newton and Gottfried Wihelm Leibniz independently discovered the calculus in the seventeenth century. In recognition of this remarkable discovery. John Von Neumann's (1903-1957) thought seems to worth quoting. "... the calculus was the first achievement of modern mathematics and it is difficult to overestimate its importance. I think it defines more equivocally than anything else the inception of modern mathematics, and the system of mathematical analysis, which is its logical development, still constitute the greatest technical advance in exact thinking."

Both Newton and Leibniz recognized that calculus can be regarded as the branch of mathematical study which treats change and the rate of change. They also observed the close connection between algebra and geometry, epitomized by the fact that every equation has a graph and every graph an equation.

By 1664, the young Newton became familiar with all mathematical ideas and results of his predecessors and was fully ready to discover his own. In his new analytical methods, Newton remarkably combined the ideas, results and methods of three largely separate branches of mathematics: coordinate geometry, calculus and infinite series (or more precisely, the representation of functions by infinite series). During 1664-1666, Newton developed all the basic ideas and methods in his first version dealing with the fluxional calculus. In this work, he treated variables as moving quantities changing with time and introduced the concept of velocity and acceleration at any instant of time. He then considered exposition of his ideas and results in the book *Methodus Fluxionum et Serieum Infinitarum* (*The Method of Fluxions and Infinite Series*) which was not published until 1736, after his death. In his book, Newton treated variables as flowing quantities generated with time by the continuous motion of points, lines and planes, rather than as static infinitesimal quantities as in his first version of the calculus. He defined a variable quantity x or y as the *fluent*, and its rate of change with respect to time as the *fluxion* which was denoted by $\dot{x}$ and $\dot{y}$ (the Newtonian dot notation) which is now known as the *derivative* or the *velocity*. Subsequently, he stated more clearly the fundamental problem of calculus by introducing any two variables rather than the time as the independent variable. For example, $y = x^n$ so that, in modern notation, $(dy/dx) = n\,x^{n-1}$. One of the outstanding problems of the seventeenth century was that of finding the tangent to a curve at an arbitrary point. It was solved by Newton's teacher at Cambridge University, Isaac Barrow (1630-1677). Newton developed the idea of the rate of change from the analytic point of view. He also demonstrated his ideas by examples of finding tangents to well-known plane curves including cycloid and spirial. He gave another example of a plane curve with its algebraic equation in the form

$$x^3 - ax^2 + axy - y^3 = 0, \tag{1.4.1}$$

to derive the fluxional equation

$$3x^2\dot{x} - 2a\,x\,\dot{x} + a(\dot{x}y + x\dot{y}) - 3y^2\dot{y} = 0. \tag{1.4.2}$$

This gives the slope (or gradient) of the tangent to the curve at any point (x, y) so that

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{(3x^2 - 2ax + ay)}{(3y^2 - ax)}. \tag{1.4.3}$$

In his book, Newton not only developed a general method for finding the instantaneous rate of change of one variable with respect to another, but proved that area can be found by reversing the process of finding a rate of change. For example, if the curve is $y = amx^{m-1}$, the area under the curve is $z = ax^m$. In modern notation, this result can be written as

$$\text{Area} = z = \int_0^x amx^{m-1}dx = ax^m. \tag{1.4.4}$$

He applied the method to calculate the area of many plane curves. Since areas can be computed by the summation of infinitesimal areas, thus the summations (or more precisely, the limits of sums) can be obtained by reversing the process of differentiation. This is now known as the *fundamental theorem of integral calculus*. He also derived the correct formula for the curvature of a given curve, namely,

$$\kappa = \frac{\ddot{y}}{(1+\dot{y}^2)^{3/2}}. \tag{1.4.5}$$

Newton obtained the area under the curve $(dz/dx) = (1+x^2)^{-1}$ in terms of Gregory's infinite series as

$$z = \tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots . \tag{1.4.6}$$

Newton's major idea of infinite series dealt with the discovery of the general binomial theorem as well as the binomial infinite series which was then applied to solve the problems of calculus. The study of infinite series led to another very important method in mathematics which deals with the solution of problems by means of *successive approximations*. This means that we first find an approximate solution of a problem, then based on this, we look for a still better solution, or a second approximation; and so on, each time getting a little better result to the exact solution. This process can be continued to find the best approximate solution. For example, if $f(x)$ is a continuous function, naturally $f(x+h)$ is approximately equal to $f(x)$ if h is small. This implies that $f(x)$ is a first approximation to $f(x+h)$, and we may write this as

$$f(x+h) = f(x) + \cdots , \tag{1.4.7}$$

where $+\cdots$ means that there is still something lacking to the exact solution. To obtain a second linear approximation, we use the definition of the first derivative, that is, $f'(x)$ is the limit of

$$\frac{f(x+h)-f(x)}{h} \quad \text{as} \quad h \to 0. \tag{1.4.8}$$

Consequently, $f(x+h) - f(x)$ is approximately equal to $h\, f'(x)$ so the next linear approximation (differentiation as *linear approximation*) is

$$f(x+h) = f(x) + h\, f'(x) + \cdots , \tag{1.4.9}$$

where $+\cdots$ has the same meaning as before. Continuing the process, we obtain the third approximation formula

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \cdots . \tag{1.4.10}$$

Thus, it is possible to continue this process so as to obtain still further closer approximations. Eventually, this leads to the celebrated formula known as *Taylor's series expansion* of $f(x+h)$ in the form

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + \cdots , \tag{1.4.11}$$

where $f^{(n)}(x)$ is the nth derivative of $f(x)$. Brook Taylor (1685-1731), a British mathematician knew this formula in 1712 without any rigorous proof, but his name has become inseparably associated with the formula. However, the Taylor series was known to another British mathematician, James Gregory (1638-1675) in 1670. Indeed the Taylor formula (1.4.11) originated from the Gregory–Newton interpolation formula for the calculus of finite differences involved in simple and elementary functions, it was apparently not discovered by Newton who, of course, knew some special cases of it. Putting $x = 0$ in (1.4.11) and replacing h by x leads to the famous Maclaurin series

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots . \tag{1.4.12}$$

This was deduced by Colin Maclaurin (1698-1746) who gave this special case in his *Treatise of Fluxions* published in 1742 and stated that it was a special case of Taylor's result (1.4.11). In his Treatise, Maclaurin made a serious attempt to establish the Newton's calculus more rigorous. It was a commendable effort but not successful due to the lack of convergence. Maclaurin succeeded James Gregory as professor of mathematics at the University of Edinburgh. On the other hand, Gregory discovered another simple but important infinite series for $y = \tan^{-1} x$ as

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots . \tag{1.4.13}$$

This is universally known as *Gregory's series*. The formula for π known as Gregory's formula is the special case of (1.4.13) by taking $x = 1$ so that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots . \tag{1.4.14}$$

This series was also discovered by Leibniz in 1674. Many mathematicians including Newton, Leibniz, Gregory, Maclaurin and Euler were interested in infinite series. One of the major uses of infinite series beyond their service in differentiation and integration was to find the series expansion of functions such as trigonometric functions, exponential and logarithmic functions, and π and e.

In his student days, Newton had made an extensive study of Descartes *La Geometrie* which prepared him well to pursue advanced study and research in geometry. In the late 1660s, he embarked on an extensive research in algebraic equations and the method of coordinate geometry. Using the method of coordinate transformations, he described fairly general geometrical curves with many examples of cubic curves. He proved that the general cubic equation containing ten terms can be expressed in the simpler form

$$axy^2 + bx^3 + cx^2 + dy + ex + f = 0. \tag{1.4.15}$$

In about 1690, Newton revised and expanded his earlier work on geometry concerned with a large number of general theorems about cubic curves and their asymptotes, and the associated conic sections. According to some mathematical historians, it seemed that Newton had a comprehensive plan to write three-volume treatise on the *Geometria*, but his work was never completed.

Sir Isaac Newton's discovery of the calculus and the fundamental mathematical and physical laws were published in his first book of *Philosophiae Naturalis Principia Mathematica* (*Mathematical Principles of Natural Philosophy*) which is considered one of the greatest single contribution ever published in the history of physical sciences. This celebrated volume, usually called *Principia* or *Principia Mathematica* was completed over three hundred years ago and communicated to the Royal Society in the Spring of 1686 and then published in 1687. In it Newton not only put forward a new theory of how bodies move in space and time, but also developed the complicated mathematics needed to analyze these motions. In addition, he also profoundly formulated the laws of motion and a law of universal gravitation according to which each body in the universe was attracted toward every other body by a force that was stronger when bodies are more massive and close to each other. It was exactly the same force that caused objects to fall to the ground. According to his law, gravity causes the Moon to move in an elliptic orbit around the Earth and the planets to follow elliptical paths around the Sun. It was the first book by Newton to contain a unified system of scientific principles explaining what happens on the Earth and the Universe.

From the time of the publication of the Newton's *Principia* and throughout the eighteenth century, the Newtonian world-view was the remarkable influence on the British intellectual life, especially, in the fields of mathematics, physics, astronomy and natural philosophy. It may be appropriate to mention at least four British mathematical scientists including John Wallis who was Newton's contemporary and one of his closest friends, and Isaac Barrow – Newton's teacher and friend. Two other younger colleagues and friends of Newton were Edmond Halley (1656-1742) and Roger Cotes (1682-1716) who were closely involved in the editing and publication of Newton's greatest work. Wallis' research on algebra and his two major books entitled *Tractatus de sectionibus conicas and Arithmetica Infinitorum* have had tremendous influence on Newton's discovery of the general binomial theorem as well as the calculus. It was equally remarkable that Newton's teacher Barrow was fully responsible for providing adequate training and help so that Newton can become the great mathematical scientist of all time. Indeed, Barrow provided numerous and generous help to his young student and friend in many different ways. For example, Barrow became the first Lucasian Chair of Mathematics at Cambridge University in 1663. In 1669, he suddenly resigned his Lucasian Chair and encouraged the University to offer this prestigious Chair to young Newton as Barrow strongly believed that Newton was the most outstanding mathematical scientist for this position.

In his independent discovery of the calculus, Leibniz began with a fairly general approach to infinitely small increments in x and y, where δx is used to indicate the difference of two infinitely close values of x, and δy to indicate the difference of two infinitely close values of y. The limiting value of the ratio $\left(\frac{\delta y}{\delta x}\right)$ as δx tends to zero is written as $\frac{dy}{dx}$ or sometimes as $D_x y$ or Dy, and is called the *derivative* of y at a point x of the curve $y = f(x)$. Geometrically, $\frac{dy}{dx} = \tan\theta$ represents the slope of the tangent to the curve $y = f(x)$ at x, where the tangent at the point x makes an angle θ with the positive direction of the x-axis. If $y = f(x)$, then its derivative of y at x is often written as $f'(x)$. Similarly, the derivative of a derivative, that is, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is written as $\frac{d^2y}{dx^2}$ or $f''(x)$ is called the *second derivative.* (The prime notation or the symbol D is due to Leibniz.) The process can be continued to give the nth derivative written as $D^n y = \frac{d^n y}{dx^n}$ or $f^{(n)}(x)$. So, in his discovery of the calculus, Leibniz first introduced the idea of symbolic method and used the symbol $\frac{d^n y}{dx^n} = D^n y$ for the nth derivative, where n is a non-negative integer. L' Hospital (1661-1704), student of Johann Bernoulli (1667-1748), asked Leibniz about the possibility that n is a fraction, "What if $n = \frac{1}{2}$?". In 1665 Leibniz replied, "It will lead to a paradox." But he added prophetically, "From this apparent paradox, one day useful consequences will be drawn." Unlike Newton, Leibniz was more concerned with operational formulas to develop the ideas, methods and results of the calculus in the broad sense. For example, Leibniz proved the formulas for the derivative of a product or quotient of two (or more) functions. His formula for $D^n(uv)$, where u and v are functions of x, and a table of integrals provided the basic rules and formulas in calculus. Newton used his fundamental concept of the fluxion to solve the problems of area and volume. According to Newton, the fundamental theorem of calculus is a direct consequence of his definition of integration, that is, the fluent can be calculated from the fluxion. Indeed, he created the infinitesimal calculus and first formulated the ideas of fluxions and fluent as early as 1664-1666, and soon developed it into a general operational method. On the other hand, Leibniz first introduced the concept of integration as summation, so his definition does not imply the fundamental theorem of integral calculus which has to be proved. This idea led him to formulate the general problem of finding the area of the curve $y = f(x)$ between the portion of the x-axis from $x = a$ and $x = b$ and on the left and right by two straight line parallel to the y-axis. We divide the x-axis into n equal subintervals so that each of the subintervals along the x-axis is δx which is the base of every one of the small rectangular areas. The height of the average rectangle is represented by a perpendicular line drawn from a typical interior point of the interval δx to the curve. Its value is, of course, $f(x)$. The area of each such average rectangle is $f(x) \cdot \delta x$, and the sum $\sum f(x)\delta(x)$ of these many small elements of area is called the *definite integral* of the function $y = f(x)$ between the values of $x = a$ and $x = b$. In the notation of Leibniz, the limiting value of the sum representing the total area A is equal to the definite integral in the form

$$A = \int_a^b f(x)dx, \tag{1.4.16}$$

where in the above sum, δx is replaced by dx and the sum by the integral as $\delta x \to 0$. Although he was in possession of his fundamental ideas, methods and formulas from 1670 onwards, Leibniz discovered the differential and integral calculus, as we have it today, during 1675 and 1685. His first paper on the subject was published in Latin in the 1684 issue of the *Acta Eruditorum Lipsienium* which was the first monthly scientific journal in Germany founded by Leibniz in 1682. So, his discovery of calculus soon became widely known in Europe, largely due to voluminous work of the Bernoulli brothers, Jakob and Johann, published in the *Acta* in the 1690s. Following the notations and results of Leibniz, the first textbook on the differential calculus, *Analyse des infiniment petits* was published by L' Hospital in 1696.

It is clear from the discovery that the problem of the area is a typical problem of the integral calculus, but there are many other major problems such as the length of a given curve, the volume of a solid body of a given shape, and the area of its curved surface. A typical problem in mechanics is: How far will a particle moving with a given law of velocity go in a given time?

Apparently, the differential and integral calculus seemed to be quite different and independent subjects. Indeed, there is a very close relationship between them. We now look at the formula (1.4.16) of the area very closely. The area of a curve with one curved side $y = f(x)$ and the base at $x = 0$ and $x = x$ is

$$F(x) = \int_0^x f(x)dx, \tag{1.4.17}$$

where the upper limit of the integral is an arbitrary but fixed number, so that the area $F(x)$ will depend on x. The natural question is: What is the derivative of $F(x)$? According to the rule of derivative, we can write

$$\frac{F(x+h) - F(x)}{h} \tag{1.4.18}$$

and proceed to the limit as $h \to 0$. Obviously, $F(x + h) - F(x)$ is the area between the curved boundary, the x-axis and the vertical lines $y = x$ and $y = x + h$. If h is very small, an appropriate figure reveals that this area $h \cdot f(x)$ so that

$$\frac{d}{dx} F(x) = f(x), \tag{1.4.19}$$

that is, the derivative of the integral is equal to the value, at the upper limit of integration, of the integrand. Differentiation of the integral naturally leads to the original function $y = f(x)$. In other words, differentiation is the inverse process of integration. This is a very major discovery, because it is usually very much easier to do differentiation than integration.

In integral calculus, many problems involve the integration of $(1/x)$ and the integral of this function is $\log_e x = \ln x$. Thus, the number e arises naturally as the base of logarithms. More precisely,

$$\int_1^x \frac{1}{x}\, dx = \log_e x. \tag{1.4.20}$$

If $y = \log_e x$, then $x = e^y$ so that $\frac{dy}{dx} = \frac{1}{x}$. Sometime, formula (1.4.20) is used as the definition of a logarithm. If this is made the starting point, the exponential function appears as the inverse of the logarithm function and vice versa.

However, both Newton and Leibniz soon recognized some major logical difficulties associated with the concept of infinitely small quantities and the limiting value of the ratio of two such small quantities. It was the celebrated French mathematician, Jean d'Alembert (1717-1783) who also recognized the extraordinary power and usefulness of the calculus in finding the solution of real-world problems. At the same time, he realized the lack of rigor in the calculus, and made a serious attempt to revive the concept of *limit* in order to give the logical foundation of the subject. In his famous Encyclopedia article on *Différentiel* published in 1754, d'Alembert used his own limit concept to explain and justify the rules of differential calculus. He presented the familiar chord and tangent figure and states that: "The differentiation of equations consists simply in finding the limit of the ratio of the finite differences of two variables included in the equation". This was essentially no more than a reformulation of the ultimate ratios of his predecessors. In 1768, d'Alembert published a short exposition of his ideas entitled *Sur les principes métaphysiques du calcul infinitésimal.* This elegantly written article served as a mathematical model of clarity and logical proof. Its objective, to quote the closing sentence, was to "provide a sufficient introduction to the subject for those who merely wish to have a general, but correct, idea of its principles." It was the nineteenth century refinement of the fundamental idea of limit that eventually resolved the basic problems of the calculus. The mathematical foundations of calculus was then firmly established in the first part of the nineteenth century through the basic concepts of analysis such as function, continuity, limit, differentiability, integrability and convergence notably due to Augustin-Louis Cauchy (1789-1857) who was considered as one of the greatest mathematical scientists in terms of rigor, clarity, elegance and generality. He is also regarded as the father of the theory of functions of a complex variable and the theory of mathematical elasticity.

Although Newton first discovered calculus in 1664-1666, and communicated his ideas and results through manuscripts and letters to selected friends from 1666 onwards, however, he never published his manuscripts during 1664-1686. In his two letters addressed to Leibniz in 1676, Newton made no mention of his 1671 manuscript "*Treatise of the method of series and fluxions*" which contained algorithms and rules of differential calculus (similar to those of Leibniz) and their applications to problems of tangents and curvatures of plane curves. Ultimately, his work on calculus was first published in his *Principia* in 1687. In the years between the publication of the second edition of *Principia* in 1712 and the death of Leibniz four years later, there had been a bitter controversy between the supporters of Newton and those of Leibniz over the priority of the discovery of the calculus. As President of the Royal Society, Newton himself participated in the priority dispute and claimed that he discovered calculus before Leibniz. However, it was soon realized that Newton's notations, the terms *fluent* and *fluxion* were far inferior to Leibniz's elegant symbolism, the concepts of differentiation and integration. Subsequently, mathematicians began using Leibniz's notations, the term *integral* instead of *fluent*, and *derivative* instead of *fluxion.* So, the Newtonian terminology became almost obsolute in literature. In his 700-page long book on calculus published in 1797, S. F. Lacroix (1765-1843) expressed his views on calculus as: "... The school of Leibniz had a marked superiority over that of Newton, due perhaps more to the superiority of the former's methods than to the genius of his disciples, the Bernoulli's .... When Newton's writing were circulated on the Continent, one could see that he was in possession of the method of fluxions well before Leibniz had invented his differential calculus; but while it was possible for Newton's genius to deduce everything from his method that Leibniz could deduce from his own, the latter could be applied much more easily than the former." At any rate, Newton might have avoided the priority dispute of calculus with Leibniz if he had published his work immediately after its completion in 1666. We now close this section with a special tribute to both great men, Newton and Leibniz , for their independent discovery of the calculus which, indeed, was the greatest intellectual achievement in the history of mathematical sciences.

The greatest achievement of the seventeenth century mathematics was calculus which, next to number theory, algebra, analytical geometry, and projective geometry, is the greatest creation in all of mathematical sciences. In addition, the methodology of modern science, Newtonian mechanics, the universal law of gravitation, astronomy and celestial mechanics have been around for some decades, and a wide variety of specific problems have been solved by new and ingenious methods. All these marked the beginning of the golden age of modern and useful mathematics and science. Fully equipped with an enormous amount of knowledge and information, Leonhard Euler appeared as the universal genius at the center stage of eighteenth century mathematics and became ready to discover, unify and disseminate scientific and mathematical knowledge.

Chapter 2

Brief Biographical Sketch and Career of Leonhard Euler

"Since a general solution must be judged impossible from want of analysis, we must be content with the knowledge of some special cases, and that all the more, since the development of various cases seems to be the only way to bringing us at last to a more perfect knowledge."

Leonhard Euler

"Read Euler, read Euler, he is the master of us all."

P. S. Laplace

"... the study of Euler's works will remain the best for different fields of mathematics and nothing else can replace it."

Friedrich Gauss

2.1 Euler's Early Life

The old city of Basel in Switzerland has many great historical traditions. It has Switzerland's tallest building, *Basler Messeturm*: a beautiful and historic Basel landmark. It is renowned for the sixteenth century *Basel Musseums* which contain a diverse and wide spectrum of collections of fine arts and represent the oldest collection in continuous existence. It is the home of the oldest university of Switzerland, the University of Basel and became very famous through the fame of extraordinary contributions of three generations of the Bernoulli family. It was the birthplace of Leonhard Euler who was born in April 15, 1707. His father, Paul Euler (1670-1745) was a Calvinist priest and a gifted amateur mathematician. Paul Euler was a student of Jakob Bernoulli (1654-1705) who was well known for his work on calculus and probability theory. Euler's mother, Margarete Brucker was also the daughter of a minister. Soon after Leonhard's birth, his father moved the family to a nearby village of Riehen, where Leonhard spent his childhood there and grew up with two younger sisters. Paul Euler wished his son to become a priest in the village church and so his son's early education and training focused on theology and related subjects. At the advice of his father, Leonhard entered the University of Basel to study theology and Hebrew. In 1720, at the age of thirteen, Euler graduated from the University with philosophy major. Fortunately, Euler's early extraordinary ability in mathematics and physics were recognized by Johann Bernoulli (1667-1748), then professor of mathematics at the University of Basel and the greatest mathematician of the time, who gave him a private lesson in mathematics in every Saturday afternoon. As Leonhard remembered:

"I was given permission to visit [Johann Bernoulli] freely every Saturday afternoon and he kindly explained to me everything I could not understand."

Leonhard soon became a close friend of his tutor's two sons, Daniel and Nicholas, and it was under their influence at college that Euler achieved his universal reputation. Johann Bernoulli soon realized that Leonhard would become a great mathematician and convinced Euler's father to allow his son to change to mathematics. So, Leonhard's father reluctantly abandoned theological ambitions for his exceptionally gifted son. However, Leonhard's early training and parential influence had struck a deep chord and so he remained a devout Calvinist all his life. As we will soon see, Euler's destiny as an universal mathematician, his life and career path were closely linked to the Bernoulli family. Once Euler himself expressed his deep gratitude to Johann Bernoulli by saying:

"I soon found an opportunity to gain introduction to the famous professor Johann Bernoulli, whose good pleasure it was to advance me further in the mathematical sciences. True, because of his business he flatly refused me private lessons, but he gave me much wiser advice, namely to get some more difficult mathematical books and work through them with all industry, and whenever I should find some check or difficulties, he gave me free access to him every Saturday afternoon and was so kind as to elucidate all difficulties, which happened with such greatly desired advantage that whenever he had obviated one check for me, because of that ten others disappeared right away, which is certainly the way to make a happy advance in the mathematical sciences."

At the age of 13, Leonhard enrolled in the department of Arts at the

University of Basel to receive a general education and training. In 1724, at the age of 17, Leonhard received his Master's degree after writing a thesis comparing the natural philosophy of René Descartes with that of Isaac Newton, and then began his independent studies and research in mathematics. At the age of nineteen, he submitted two dissertations to the Paris Academy of Science, one on the masting of ships, and the other on science of sound. This work marked the beginning of his splendid research career in mathematics and science.

2.2 Euler's Professional Career

Euler's first research paper on the construction of isochronous curves in a resisting medium, and his second paper on method of finding reciprocal algebraic trajectories were published in 1726 and 1727 respectively in the international journal *Acta Eruditorum.* In 1727, Euler submitted a memoir to win the prize of the Paris Academy concerning the masting of sailing ships and although he was unsuccessful, he received an honorable mention. However, subsequently, he won the same prize at least twelve times. With the encouragement of Johann Bernoulli, Euler applied for the vacant professorship of physics at the University of Basel, but he did not get the position, partly because he was too young for such a high position. After the establishment of the Royal Academy of Sciences at St. Petersburg by the emperor Peter the Great, the two eldest sons of Johann Bernoulli were invited to join the Academy in 1725. In the autumn of 1725, Johann's two sons, Nicholas (1695-1726) and Daniel (1700-1782) went to St. Petersburg, Russia. Euler maintained regular contact with them in St. Petersburg. Unfortunately, Nicholas was drowned in July of 1726. Daniel Bernoulli joined the newly established Imperial Russian Academy of Science in St. Petersburg. With the support of Daniel, Euler received an offer from the Russian Academy for the position of Medical Associate, and then left his motherland to join this new job at the St. Petersburg Academy in 1727. In a few months, he managed to get transferred to the mathematics-physics section of the Academy as permanent member in 1727. He continued to conduct his research with Daniel Bernoulli mainly in mechanics and physics, particularly, in hydrodynamics. In St. Petersburg, Euler was surrounded by a group of famous mathematical scientists including Daniel Bernoulli, an applied mathematician, and Jacob Hermann, an analyst and geometer. They made the unique contribution to the flowering of Euler's mathematical genius. Although Euler's genius manifested itself in his early work, his indebtedness to classical and contemporary mathematical scientists was also more plainly evident than in his later contributions.

In 1733, Daniel returned to his home country, Switzerland to occupy the prestigious Basel Chair of Mathematics. At the age of 26, Euler was selected to serve as the leading mathematical position in the Academy. Christian Goldback (1690-1764) and Euler began to make correspondence in 1729 by letters on mathematical problems including algebra and number theory. Indeed, Euler's enthusiasm and interest in algebra, geometry, geometrical optics, and number theory originated from Goldback's correspondence which he continued until Goldback's death in 1764. Goldback was a kind of academic mentor for Euler, and it was he who introduced Euler to number theory and geometry through the works of Pierre de Fermat. During his fourteen-year stay from 1727 to 1741 in St. Petersburg, Euler published a very large amount of new and original research in mathematics and physics, and also wrote many elementary and advanced books on mathematics. One of the major unsolved problems of the time was the so called *Basel problem* (after the name of the City of Basel) which was formulated by Pietro Mengoli (1625-1686) in 1644. Both famous Jakob and Johann Bernoulli brothers came from the City of Basel, and they made serious but unsuccessful attempts to solve this classical problem. During his stay at St. Petersburg, among his many remarkable discoveries, three mathematical formulas can be quoted as an epitome of what Euler discovered: One was his renowned solution of the Basel problem of finding the sum of the squares of the reciprocals of the integers, that is,

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2.1}$$

Euler's work on the famous zeta function $\zeta(s)$ for real s defined by the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2.2.2}$$

began around 1730 with his first discovery of the value $\zeta(2) = (\pi^2/6)$ in 1735 and he continued his research for the value $\zeta(2n)$ for the natural number $n \geq 1$. Aigner and Ziegler (2001) reported two proofs of the Basel problem and Chapman (1999) collected fourteen proofs of the problem.

Almost 110 years before Bernhard Riemann's (1826-1866) discovery of $\zeta(s)$ for complex s, using the summation of divergent series and mathematical induction, Euler in 1749 discovered a remarkable functional equation for the zeta function in the form

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{2.2.3}$$

where $\Gamma(s)$ is the Euler gamma function which was also invented by Euler in 1729 as a generalization of the factorial function. When $s = 1$, the value $\zeta(1)$ led him to discover the universally known as the *divergent harmonic series* (as any term in it is the harmonic mean of two neighboring terms)

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n}. \tag{2.2.4}$$

His study of the harmonic series enabled him to establish two more remarkable results. The first one was the discovery of the new mathematical constant which is now known as the *Euler universal constant* γ given by

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log_e n\right) = 0.577215665... \tag{2.2.5}$$

Euler calculated this constant up to 16 decimal places. Although the numerical value of γ is known today to hundreds of decimal places. However, it is not known even today whether γ is rational or irrational.

The second result dealt with wonderful and unexpected connection between number theory and analysis. Using a fairly simple argument, Euler proved in 1737 that the divergence of the harmonic series implies that the number of primes is infinite and vice-versa. In modern notation, this result reads

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p}\left(1 - \frac{1}{p}\right)^{-1}, \tag{2.2.6}$$

where the product is taken over all primes p. However, this is an invalid identity as Euler paid no attention to convergence of this harmonic series and the infinite product in (2.2.6). A few years later, Euler generalized his identity (2.2.6) and discovered another remarkable identity that expresses the zeta function as an infinite product extended over prime numbers only. More precisely, he proved a strikingly new theorem, for $s > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p}\left(1 - \frac{1}{p^s}\right)^{-1}, \tag{2.2.7}$$

where the infinite product in taken over all prime numbers p. Thus, Euler's theorem clearly demonstrates that the zeta function plays a fundamental role in number theory and laid the modern foundations of the analytic number theory. More remarkably, the Euler theorem may be considered as an analytical statement of the *Fundamental Theorem of Arithmetic* which states that any natural number greater than one can be uniquely expressed as a product of primes.

Two of his greatest discoveries were the *Euler formulas*

$$e^{\pm ix} = \cos x \pm i \sin x. \tag{2.2.8}$$

In particular, $x = \pi$ and 2π led to the legendary formulas

$$e^{i\pi} = -1 \quad \text{or} \quad e^{i\pi} + 1 = 0 \quad \text{and} \quad e^{2\pi i} - 1 = 0. \tag{2.2.9ab}$$

These simple formulas beautifully and unexpectedly relate six most fundamental constants e, i, π, 0, 1 and -1 in mathematics and science. Before Euler, a number of great mathematicians made an attempt to establish exact formulas for the computation of the transcendental numbers e and π. In retrospect, Euler first discovered power series representations and continued fraction expansions of these numbers and developed a clear and original treatment of using methods of analysis in order to understand the properties of these numbers.

Of particular interest is Euler's remarkable work on the formulation of many problems in solid and fluid mechanics in mathematical language and his development of analytical methods of solving these mathematical problems. Once Joseph Louis Lagrange (1736-1813) made a delightful statement about Euler's work in mechanics, "The first great work in which analysis is applied to the science of movement." More remarkable was Euler's extensive study of ordinary and partial differential equations in his work in mechanics and mathematical physics. He first introduced the concept of an integrating factor and gave a general treatment of linear ordinary differential equations with constant and variable coefficients in 1739 with a careful distinction between complimentary function, particular and general solutions. He also considered the possibility of reducing second order equations to first order equations by a suitable change of variables. He also made some major contributions to the method of power series solutions, and first developed the technique for approximating the solution of the equation $\frac{dy}{dx} = f(x, y)$ with initial conditions $x = x_0$ and $y = y_0$ and its extension to second-order differential equations.

Considerable attention to the second-order partial differential equations was given by many great mathematical scientists including, Euler, Daniel Bernoulli, Jean d'Alembert, Lagrange, and Joseph Fourier (1768-1830). Some frequently occuring equations in applied mathematics and physics include the wave equation, the potential (Laplace) equation and the diffusion

(Fourier) equation. The classic wave equation, or a generalization of it, almost inevitably occurs in any mathematical study of physical phenomenon involving the propagation of waves in a continuous medium. For example, the mathematical analysis of sound waves, water waves, shock waves, and electromagnetic waves are all based on this equation. Among others, Euler made some major contributions to the solution of wave equation and its physical features in 1750. He also first derived the Laplace equation in 1752 in his study of hydrodynamics. His work on vibrating elastic membranes led him to the famous Bessel equations which he solved in terms of Bessel functions. He also made important contributions to the theory of Fourier series and gave the first systematic treatment of the calculus of variations.

It was Euler who invented a series of mathematical symbols including the natural logarithmic base e, the imaginary quantity $i = \sqrt{-1}$, the symbol Δ representing a finite difference, and Σ denoting summation. These symbols have universally been adopted in mathematical sciences. In addition, he first introduced a large number of notations including $f(x)$ for a function of x, and first treated e^x, $\sin x$, $\cos x$, $\log x$ as functions of x in calculus and analysis, A, B, C and a, b, c for the angles and sides of a triangle in geometry and trigonometry. Even today, all of his notations and symbols are true historic landmarks in mathematics and science and are still considered as a unique mark of his genius. In his 1748 two-volume remarkable treatise on *Introductio in analysin infinitorum*, Euler first introduced the concept of a function as a correspondence of values and then gave the following working definition:

"A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quality and numbers or constant quantities."

Although this is not a precise and modern definition of a function, Euler continuously used his definition to study polynomials, exponential, trigonometric and logarithm functions and their properties. In his *Introductio*, he subsequently recognized functions as the fundamental building blocks of real and complex analysis. He then considered mathematical analysis as the study of algebraic, trigonometric, exponential and transcendental functions facilitated by differentiation and integration. After the discovery of the imaginary number i, Euler independently recognized the need for a more extensive investigation of complex numbers in the form $a + ib$. In 1749, Euler made an attempt to give a first unsatisfactory proof of the *Fundamental Theorem of Algebra* which states that an nth degree polynomial equation $f(z) = 0$ with real or complex coefficients has at least one root, real or complex, where $z = x + iy$ is a complex number. Using the method of functions of a complex variable, Euler first successfully derived the Cauchy–Riemann equations

$$u_x = v_y, \qquad v_x = -u_y, \tag{2.2.10}$$

where $f(z) = u(x, y) + iv(x, y)$ is a complex function of a complex variable z. Almost simultaneously, he successfully applied these equations to investigate the problems of fluid and solid mechanics. These equations became the most fundamental basis for the subsequent rigorous development of the theory of functions of a complex variable primarily due to Friedrich Gauss (1777-1855), Augustin-Louis Cauchy, Karl Weierstrass (1815-1897) and Bernhard Riemann during the nineteenth century. Stimulated by the work of Johann Bernoulli on orthogonal trajectories of a family of curves in 1698, Euler continued his research further on this topic which led to the beginnings of the theory of conformal mappings in complex analysis. The power and beauty of conformal mappings became a powerful tool for the solutions of problems in fluid mechanics, heat conduction and electromagnetic theory in the late 1800s.

By the year 1730, Euler had already achieved a considerable reputation as a pure and applied mathematician. Many of his contemporaries thought of his early work as his most significant mathematical achievement. In the same year, Euler was selected to serve as Professor of Physics at St. Petersburg Academy and in 1733, he succeeded his close friend Daniel Bernoulli as the Chair of Mathematics, and became in charge of the Geography Department, where he was actively involved in cartographic research work jointly with the well-known French astronomer and geographer, J. N. Delisle (1688-1768) who was invited by the Russian czar Peter the Great to St. Petersburg Academy to create and run the school of astronomy. Because of his secured job in the Academy and his great reputation as the leading mathematician in Europe, Euler decided to settle in Russia and then married Catharina Gsell (1707-1773) in 1734, the daughter of a Swiss artist and painter then working in Russia. They had thirteen children – eight died in infancy, and only three sons, Johann-Albrecht (1734-1800), Carl (1740-1790) and Christopher (1743-1808) and two daughters, Catherine-Elena (1741-1781) and Charlotte (1744-1780) survived.

At the tenth anniversary of his arrival in St. Petersburg, Euler wrote a letter to the President of the St. Petersburg Academy of Sciences in 1737 indicating his major job responsibilities as follows:

"According to my conditions of service in the Imperial Academy of Sciences, I am obliged to fulfil the following:

1. To attend the meetings of the Conference, which I fulfil assiduously and always have in readiness articles to read there.

2. To give lectures to students on the higher branches of mathematics. This also, whenever students wanting to study that subject present themselves, I carry out according to their capabilities.

3. I have also been commissioned to participate in the work on the geography of Russia, and here I also work as far as my strength and my duties allow.

As far as my other labors are concerned at present, and also in the future, I am now working on an arithmetic to be used in the Academy's gymnasium. Apart from that, I have the intention, if my other activities do not interfere, to bring to completion several works in hand, having to do with music, statics, the analysis of infinities, and the motion of bodies in water."

It is evident from this letter that research, teaching and service have been his major job responsibilities three hundred years ago as many universities professors do regularly even today. However, Euler's teaching load was relatively light at the Academy, but his research contributions far exceeded the normal mathematician by any standard. He quickly became the legendary figure in the world of mathematical sciences, both on the discovery of new knowledge and its dissemination through publications, presentations and writings.

Euler was extremely proficient in many languages, especially in Latin, French, German and Russian as he efficiently and effectively used these languages in writing research papers, books and correspondence. He was extremely knowledgeable about the ancient history of mathematics and science and had phenomenal memory of historical events and people. Amazingly, he had unusual knowledge of other subjects including botany, chemistry and medicine, even though he did not work on these subjects.

Perhaps, among other things, Euler's best known research in this period was his original formulation and solution of the famous problem of the Seven Bridges of the Königsberg in 1736. This marked the beginning of a new area of mathematics known today as graph theory. In 1736, Euler published his two-volume large treatise *Mechanica sive motus scientia analytice exposita (Mechanics or the Science of Motion, expounded analytically).* This work dealt with a comprehensive treatment of almost all aspects of analytical mechanics including the mechanics of particles, rigid, flexible and elastic bodies as well as fluid dynamics, celestical mechanics and ballistics. This work led him to formulate the basic formulation of mechanics, in general and Newtonian mechanics, in particular. After winning the Grand Prix of the *Paris Academy* in 1738 and 1740, Euler became an eminent mathematical scientist in the whole Europe. Although he was always active and productive in mathematical sciences, his main relaxation was music. He was then actively involved with advanced study and research in the theory of music and publication of his treatise *An attempt at a new theory of music, clearly expounded on the most reliable principles of harmony* in 1739, and several music-theoretical manuscripts during 1766-1768. Evidently, Euler's music-theoretical works were a modest part of his total mathematical and scientific legacy. It was during this first St. Petersburg period that he became blind in the right eye in 1738.

Unfortunately, in 1740s the political conditions of Russia became very unstable as the Russian government passes into the hands of those who did not want to provide sufficient support for scientific research. Euler became very unhappy and concerned about his future in the St. Petersburg Academy. In the meantime, Frederick the Great of Prussia succeeded to the Prussian throne in June 1740, and invited Euler to serve as Director of the Mathematics Section of the newly organized Berlin Academy of Sciences (originally founded by G. W. Leibniz in 1700) with a complete academic freedom of research. In 1741, Euler left St. Petersburg to join the position at the Berlin Academy as a world-renowned 34-year old research mathematician, although he continued to receive pension from the St Petersburg Academy. While maintaining regular professional contacts and correspondence with the St. Petersburg Academy, he remained in Berlin for a period of 25 years from 1741 to 1766 and completed his greatest work there in different fields of physics, mechanics, pure and applied mathematics. During his stay in Berlin, he also completed his 1744 masterpiece, the memoir on the calculus of variations, known as *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes sive solutio problematis isoperimatrici latissimo sensu acceptl (A method for discovering curved lines that enjoy a maximum or minimum property, or the solution of the isoperimetric problem taken in its widest sense).* This effectively created a new branch of mathematics which is known today as the *Calculus of Variations*. Its publication in 1744 led to his election to the Royal Society of London and to the Paris Academy, among many other honors and awards. His theorological conviction led him to believe that all natural phenomena operate in such a way that some combination of physical variables is either minimized or maximized, and then he made the following delightful statement:

"For since the fabric of the Universe is most perfect and the work of a most wise Creator, nothing at all takes place in the Universe in which some rule of maximum or minimum does not appear."

At the end of the seventeenth century, many important questions and problems in geometry and mechanics involved minimizing or maximizing of certain integrals for two reasons. The first of these were several existence problems, such as, Newton's problem of missile of least resistance, Bernoulli's isoperimetric problem, Bernoulli's problem of the brachistochrone (brachistos means shortest, chronos means time), the problem of minimal surfaces due to Joseph Plateau (1801-1883), and Fermat's principle of least time. Indeed, the variational principle as applied to the propagation and reflection of light in a medium was first enunciated in 1662 by one of the greatest mathematicians of the seventeenth century, Pierre de Fermat. According to his principle, a ray of light travels in a homogeneous medium from one point to another along a path in a minimum time. The second reason is somewhat philosophical, that is, how to discover a minimizing principle in nature. The following 1744 statement of Euler is characteristic of the philosophical origin of what is known as the *principle of least action* as a guiding principle in nature:

"As the construction of the universe is the most perfect possible, being the handiwork of all-wise Maker, nothing can be met with in the world in which some maximal or minimal property is not displayed. There is, consequently, no doubt but all the effects of the world can be derived by the method of maxima and minima from their final courses as well as from their efficient ones."

In the middle of the eighteenth century, a famous French scientist, Pierre de Maupertuis (1698-1759) stated a fundamental principle known as the *principle of least action*, as a guide to the nature of the universe. A still more precise and general formulation of Maupertuis' principle of least action was given by Lagrange in his Analytical Mechanics published in 1788. He formulated it as

$$\delta S = \delta \int_{t_1}^{t_2} (2T)\, dt = 0, \tag{2.2.11}$$

where T is the kinematic energy of a dynamical system with a constraint that the total energy, $(T+V)$, is constant along the trajectories and V is the potential energy of the system. He also derived the celebrated equation of motion for a holonomic dynamical system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i, \tag{2.2.12}$$

where q_i are the generalized coordinates, $\dot{q}_i$ is the generalized velocity, and Q_i is the force. For a conservative dynamical system, $Q_i = -\frac{\partial V}{\partial q_i}$, $V = V(q_i)$, $\frac{\partial V}{\partial \dot{q}_i} = 0$, then (2.2.12) can be expressed in terms of the Lagrangian, $L = T - V$, as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{2.2.13}$$

This principle was then effectively reformulated by Euler in a way that made it useful in mathematics and physics.

The work of Lagrange remained unchanged for about half a century until William R. Hamilton (1805-1865) published his research on the general method in analytical dynamics which gave a new and very appealing form to the Lagrange equations. Hamilton's work also included his own variational principle. In his work on optics during 1834-1835, Hamilton elaborated a new principle of mechanics known as *Hamilton's principle* describing the stationary action for a conservative dynamical system in the form

$$\delta A = \delta \int_{t_0}^{t_1} (T - V)\, dt = \delta \int_{t_0}^{t_1} L\, dt = 0. \tag{2.2.14}$$

Hamilton's principle (2.2.14) readily led to the Lagrange equation (2.2.12). In terms of time t, the generalized coordinates q_i, and the generalized momenta $p_i = (\partial L/\dot{q}_i)$ which characterize the state of a dynamical system, Hamilton introduced the function

$$H(q_i, p_i, t) = p_i \dot{q}_i - L(q_i, p_i, t), \tag{2.2.15}$$

and then used it to represent the equation of motion (2.2.12) as a system of first order partial differential equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial \dot{q}_i}. \tag{2.2.16}$$

These equations are known as the celebrated *Hamilton canonical equations of motion*, and the function $H(q_i, p_i, t)$ is referred to as the *Hamiltonian* which is equal to the total energy of the system. Following the work of Hamilton, Karl Gustav Jacob Jacobi (1804-1851), Mikhail Ostrogradsky (1801-1862), and Henri Poincaré (1854-1912) put forth new modifications of the variational principle. Indeed, the action integral S can be regarded as a function of generalized coordinates and time provided the terminal point is not fixed. In 1842, Jacobi showed that S satisfies the first order partial differential equation

$$\frac{\partial S}{\partial t} + H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) = 0, \tag{2.2.17}$$

which is known as the *Hamilton–Jacobi equation.* In 1892, Poincaré defined the action integral on the trajectories in phase space of the variable q_i and p_i as

$$S = \int_{t_0}^{t_1} \left[p_i \dot{q}_i - H\left(p_i, q_i\right)\right] dt, \tag{2.2.18}$$

and then formulated another modification of the Hamilton variational principle which also yields the Hamilton canonical equations (2.2.16). From (2.2.17) also follows the celebrated Poincaré–Cartan invariant

$$I = \oint_C \left(p_i \delta q_i - H \delta t\right), \tag{2.2.19}$$

where C is an arbitrary closed contour in the phase space.

Indeed, the discovery of the calculus of variations in a modern sense began with the independent work of Euler and Lagrange. The first necessary condition for the existence of an extremum of a functional in a domain leads to the celebrated *Euler–Lagrange equation.* This equation in its various forms now assumes primary importance, and more emphasis is given to the first variation mainly due to its power to produce significant equations than to the second variations which is of fundamental importance in answering the question of whether or not an extremal actually provides a minimum (or a maximum). Thus, the fundamental concepts of the calculus of variations were developed in the eighteenth century in order to obtain the differential equations of applied mathematics and mathematical physics. During its major early developments, the problems of the calculus of variations were reduced to questions of the existence of differential equations problems until David Hilbert developed a new method in which the existence of a minimizing function was established directly as the limit of a sequence of approximations.

Considerable attention has been given to the problem of finding a necessary and sufficient condition for the existence of a function which extremized the given functional. Although the problem of finding a sufficient condition is a difficult one, Legendre and Jacobi discovered a second necessary condition and a third necessary condition respectively. Finally, it was Karl Weierstrass who first provided a satisfactory foundation to the theory of calculus of variations in his lectures at Berlin between 1856 and 1890. His lectures were essentially concerned with a complete review of the work of Legendre and Jacobi. At the same time, he reexamined the concepts of the first and second variations, and looked for a sufficient condition associated with the problem. In contrast to the work of his predecessors, Weierstrass introduced the new ideas of 'strong variations' and 'the excess function' which led him to discover a fourth necessary condition and a satisfactory sufficient condition. Some of his outstanding discoveries announced in his lectures were published in his collected work. At the conclusion of his famous lecture on 'Mathematical Problems' at the Paris International Congress of Mathematicians in 1900, David Hilbert, perhaps the most brilliant mathematician of the late nineteenth century, gave a new method for the discussion of the minimum value of a functional. He obtained another derivation of Weierstrass's excess function and a new approach to Jacobi's problem of determining necessary and sufficient conditions for the existence of a minimum of a functional; all this without the use of the second variation. Finally, the calculus of variations entered the new and wider field of 'global' problems with the original work of George D. Birkhoff (1884-1944) and his associates. They succeeded in liberating the theory of calculus of variations from the limitations imposed by the restriction to 'small variations', and gave a general treatment of the global theory of the subject with large variations.

About eight years after his arrival to Frederick's court from St Petersburg, Euler received a royal assignment directly from Frederick to develop mathematics for a proposal of the Berlin lottery similar to that of the Genoese lottery in Italy. In fact, Euler started working on the probability theory related to the Genoese lottery when Frederick's letter of September 15, 1749 arrived along with a copy of a proposal for the Berlin lottery which was made by an Italian businessman named Roccolini. Euler expanded mathematical analysis of the Genoese lottery and wrote several papers on the calculus of probability with applications to mathematical games, insurance and gambling, the theory of risk, mathematical statistics involving observational error and the foundation of life insurance. There was also a correspondence between Euler and Frederick in 1763 concerning the lottery proposal, quite similar to the one of 1749. At the royal request, Euler became very interested in various aspects of the Genoese lottery system and came up with an improved lottery system. Eventually, the Berlin lottery was introduced in Germany in 1763 to raise additional revenue for the King Frederick. In addition to lottery, Euler was especially interested in more general mathematical problems of winning certain games involving coins, dice or cards. In connection with such problems, there was a paradox, known as the *St. Petersburg Paradox*. One version of this paradox can be stated as follows. A person is to toss an ideal coin until he throws a head. If he throws head at the nth throw, and not before, he must win an amount of £2^n. What is the expected value? What is the value of the game? Clearly, the probability of n heads in a row is $p(n) = \left(\frac{1}{2}\right)^n$ and the expected value is

$$\sum_{n=1}^{\infty} n\,p(n) = \sum_{n=1}^{\infty} \frac{n}{2^n} = 2,$$

and the value of the game is

$$\sum_{n=1}^{\infty} 2^n\,p(n) = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 1 = \infty.$$

Although the expected value is 2, but the value of the game is infinite so that nobody would pay an infinite amount of money to play the game.

In 1745, in response to another royal assignment by Frederick the Great, Euler translated the 1742 remarkable book on *New Principles of Gunnery* written by a great British applied mathematician, Benjamin Robins (1707-1751) who was born in the same year as Euler. His German translation of Robins' 150-page book with a large and extensive mathematical commentaries became over 700-page monumental book. Through their ordinary works, Euler and Robins revolutionized experimental, mathematical and engineering research in ballistics of the 18th and 19th centuries. Using air-resistance values based on Robins' experimental measurements, Euler obtained the solution of the equations of subsonic ballistic motion in 1753, and presented some of his numerical results into convenient ballistics tables. This was the first published Euler's analysis of projectile trajectories to incorporate empirical air-resistance values. The King Frederick was very much interested in the modern research in ballistics in order to increase the mathematical, scientific and engineering research and education of his artillery officers. Frederick the Great praised Euler's new and remarkable ballistics research effort as it provided a tremendous help for the Prussian artillery during the French Revolutionary War. In addition to Euler's original work on the theory of probability and ballistics research, he was also involved with earlier royal assignment on the design of a new hydraulic system for Frederick's summer palace, *Sans-Souci*.

Based on his many research contributions to physics and astronomy – especially celestial mechanics, Euler published his work on the theory of lunar motion in 1753 and was directly responsible for the organization of the St. Petersburg Observatory of the first ever Time Service in Russia in collaboration with J. N. Delisle. Based on his extensive use of his St. Petersburg experience, Euler was also fully responsible for the reconstruction of the Berlin Observatory with new instruments 1744 similar to those in the St. Petersburg Observatory. In collaboration with the Berlin astronomers, in 1748 Euler made observations of a ring-shaped solar ellipse in the course of which the question of the existence of a lunar atmosphere was resolved once for all — the same question was raised by Euler and his St. Petersburg scientific colleagues in 1729. These researchers and other assignments illuminate several admirable qualities of Euler's personal character: his vast mathematical and scientific imagination, his sense of duty and devotion, his tremendous research expertise to solve mathematical and scientific problems for the benefit of the society, and his remarkable ability to delight in mathematical recreation. Unfortunately, works of Euler were not fully appreciated by the King Frederick – even though the King was aware of Euler's great scientific talent and creativity.

Euler also completed his works on the theory of ships, shipbuilding and navigation which culminated in the publication of *Scientia navalis seu tractatus de construendis as dirigendis navibus* in two volumes in 1749. Walter Habicht (see Burckhardt et al. (1983)) described the fundamental importance of this work as follows:

"Following the *Mechanica sive motus scientia analytice exposita* which appeared in 1736, it [the *Scientia navalis...*] is the second milestone in the development of rational mechanics, and to this day has lost none of its importance. The principles of hydrostatics are presented here, for the first time, in complete clarity; based on them is a scientific foundation of the theory of shipbuilding. In fact, the topics treated here permit insights into all the related developments in mechanics during the eighteenth century."

In 1750, Euler discovered his famous and universal polyhedra formula

$$V - E + F = 2, \tag{2.2.20}$$

where V is the number of vertices, E is the number of edges and F is the number of faces of a regular polyhedra and made a serious attempt to prove it. In addition to his enormously busy research activities, Euler accepted numerous other duties that included supervising the observatory and botanic gardens, selecting staff, and managing budgets, calendars and maps which provided a new source of income for the Academy. In spite of Euler's tremendous contributions to the Berlin Academy in many different ways, Frederick the Great appointed one of the great French scientists, Pierre de Maupertuis as President of the Berlin Academy of Sciences in 1746. Maupertuis served the Academy as its President until his death in 1759. During his term as President, Maupertuis maintained a cordial relationship with Euler, and he once said that Euler demonstrated over many years by "his honesty, ability and zeal" that he was a most qualified and competent academic administrator of the Academy. In 1759, Euler took over the management of the Berlin Academy under the direct supervision of the King Frederick and did a magnificent job. Even though Euler was very loyal to the King, the King did not fully trust and appreciate Euler's work. When King Frederick offered the Presidency of the Academy to d'Alembert who was Euler's chief rival in scientific matters, Euler became increasingly concerned about his future in Berlin even though his 25-year stay in Berlin can be considered as the second golden period of his life.

However, in 1766, at the age of 59, Euler decided to move back to St Petersburg at the cordial invitation of Empress Catherine the Great of Russia who subsequently treated him as a visiting royalty. After his arrival at St. Petersburg with his eldest son, Johann-Albrecht in 1766, Johann was appointed professor of physics and academician of the Petersburg Academy of Sciences. With the generous support from the Russian government, Leonhard Euler was given a very large and beautiful house on the banks of the river Neva, conveniently located near the premises of the St. Petersburg Academy of Sciences. Euler's second son Carl was a medical doctor. After he returned to St. Petersburg with his father, Carl was appointed medical doctor to Empress Catherine II of Russia and a member of the State Medical Office, and from 1772 he served as a doctor of the Petersburg Academy of Sciences. Carl was also involved in research on planetary motion with his father and received a research prize from the Paris Academy of Sciences in 1760. His third son, Christopher was born in Berlin and became the Major General of artillery and Director of the Sesterbetsk factories at the court of Empress Catherine II, and was also well known for his research on astronomical observations.

In 1773, Euler invited Nickolai Ivanovich Fuss (1755-1826) to come to St. Petersburg from Basel in order to work as his major research assistant. After his arrival at St. Petersburg, Fuss lived in Euler's home for a period of ten years and became Euler's family friend and closest research assistant who faithfully helped him prepare about 355 research papers and books for publication during 1773-1783. Nickolai remained a very loyal assistant of Euler throughout his life. In view of his own outstanding research contributions to mathematics, mechanics and astronomy, Fuss was elected academician of St. Petersburg Academy in 1783 and he then married a granddaughter of Euler in 1784. He was then selected to serve as permanent secretary of the Petersburg Academy from 1800-1826.

During 1769-1771, he published his three-volume textbook on *Dioptrics*.

This work dealt with his extensive research in optical sciences and optical instruments including microscopes and telescopes. Euler's goal was to improve in many different ways optical instruments in particular telescopes and microscopes so that they could be brought to highest degree of perfection. His contributions to the fields of physical and geometrical optics was concerned with the problem of diffraction of light in the atmosphere. Some of Euler's work is best described by Walter Habicht in 1983 as stated by J. J. Burckhardt (1983):

"He began by deriving a very general differential equation; naturally, it turned out not to be integrable – it would have been a miracle had that not happened. Then he searched for conditions which make a solution possible, and finally he solved the problem in several cases under practically plausible assumptions.

Euler frequently expressed the opinion that the phenomena in optics, electricity and magnetism are closely related (as states of the ether), and that therefore they should receive simultaneous and equal treatment. This prophetic dream of Euler concerning the unity of physics could only be realized after the construction of bridges (experimental as well as theoretical) which were missing in Euler's time. These were later built by Faraday, W. Weber and Maxwell."

Euler wrote many great research treatises and textbooks of very high quality. His career involved relatively no teaching of mathematics or physics that almost many mathematical scientists have done regularly over the years. But his whole professional life was devoted to mainly research and writings at the Imperial Academy of Sciences at St. Petersburg and the Royal Academy of Sciences in Berlin. However, he seemed to have had tremendous success at writing many books and some student teaching at all levels. Perhaps, it may be appropriate to mention some of his great textbooks including *Institutiones Calculi Differentialis*, published in 1755, on differential calculus, and *Institutiones Calculi Integralis* in three volumes, published in 1768-1770, on integral calculus. Although almost all of his differential and integral calculus books were somewhat at elementary levels, the third volume of his integral calculus contained a largely expanded mathematical analysis of the calculus of variations which Euler himself discovered in 1744. In celebrating the 275th anniversary of Euler's birth in October 1983 in the Great Hall of the Moscow House of Scientists, A. P. Yushkevich delivered a lecture on "Leonhard Euler: His life and work" describing many aspects of Euler's life and voluminous contributions to mathematics and sciences and made the following concluding remark:

"Always a student with the widest interests, throughout his life he was prepared to learn from others, and in his own works often expounded other's discoveries in more convenient and accessible form. However of course overall he gave to others immeasurably more than he took from them. He influenced the work of many generations of mathematicians; in particular the St. Petersburg mathematics school of the second half of the 19th century and the first quarter of the 20th, founded by P. L. Chebyshev, was very close in spirit to Euler."

On the other hand, the academician N. I. Fuss, one of Euler's closest student and research assistant, made the following statement on Euler's writing of textbooks:

"...who had no smattering of mathematics but was the writer to whom Euler dictated his textbook *Vollständige Anleitung zur Algebra*, as generally admired for circumstances in which it was composed as for the supreme degree of clarity and of method that prevails throughout. The creative spirit reveals itself even in the purely elementary work."

In order to provide the evidence of remarkable success story of Euler as a writer and teacher, it is most appropriate to mention his famous *Letters to a German Princess*, Anahalt-Dessau, 15-year old niece of the King of Prussia on different subjects including natural philosophy, astronomy, optics, acoustics, mechanics, physics, music, electricity and magnetism. This *Letters* was one of the most remarkable and popular science books ever written in the history of science. This was written "with a marvelous clarity", according to N. I. Fuss. This *Letters* was translated from German into eight different languages including Russian, French, English, Swedish, Italian, Spanish, Dutch and Danish. This book was a compilation of over 200 different letters which were written for the instruction of the 15-year old Princess on popular subjects such as sound, light, logic, gravity, language, music and astronomy. However, it may have been written at a level well beyond that of a 15-year old. In one of his letters dated 1760 on vision, Euler began with an interesting statement:

"I am now enabled to explain the phenomena of vision, which is undoubtedly one of the greatest operations of nature that human mind can contemplate".

The publication of a Russian translation of the *Letters* in four volumes had a tremendous influence in Russia, and it served as the first encyclopedia of physics in Russia. An Englishman, Henry Hunter was the first translator of the *Letters* into English and he made the following statement in his Preface:

"It was long a matter of surprise to me, that a work so well known, and so justly esteemed, over the whole European Continent, as Euler's *Letters to a German Princess*, should never have made its way into our Island, in the language of the Country. While Petersburg, Berlin, Paris, nay the capital of every petty German principality, was profiting by the ingenious labors of this amiable man, and acute philosopher, the name of Euler was a sound unknown to the ear of youth in the British metropolis. I was mortified to reflect that the specious and seductive productions of a *Rousseau*, and the poisonous effusions of a *Voltaire*, should be in the hands of so many young men, not to say young women, to the perversion of the understanding, and the corruption of the moral principle, while the simple and useful instructions of the virtuous Euler were hardly mentioned".

At least one other scholarly work of Euler might be added to illustrate his great success as teacher and writer. This was his 1748 two-volume masterpiece treatise on mathematical analysis entitled *Introductio in analysin infinitorum* that was though not basically a textbook, but it was a remarkable compilation of large amount of material of analysis together with an extensive new material added by Euler. It was one of Euler's most brilliant and rewarding books, "as marvelous in its clarity of exposition as for the richness of its contents" according to Fuss. These represented some of Euler's prodigious contributions as writer and teacher. During his lifetime, Euler became a living legend and versatile creative genius. According to Laplace, Euler was the premier teacher of all mathematicians of his time. Lagrange, Laplace and Gauss were truly influenced by Euler's works. Once Gauss said that

"...the study of Euler's work will remain the best school for different fields of mathematics and nothing else can replace it."

On several occasions, George Polya (1887-1985) provided an extensive account of Euler's remarkable mathematical discoveries, and unique ability of presenting mathematics to a wide audience. Polya himself was a strong advocate of Euler's style of writing and teaching mathematics as he described in his 1954 book on *Mathematics & Plausible Reasoning, Induction and Analogy in Mathematics*, and he quoted M. Condorcet as saying that:

"He [Euler] preferred instructing his pupils to the little satisfaction of amazing them. He would have thought not to have done enough for science if he should have failed to add to the discoveries, with which he enriched science, the candid exposition of the ideas that led him to those discoveries."

As an advocate of Euler's clear and elegant ability of exposition of mathematics and science, Polya's quote in his 1954 book is a delight to state:

"A master of inductive research in mathematics, he [Euler] made important discoveries (on infinite series, in the Theory of Numbers, and in other branches of mathematics) by *induction*, that is, by observation, daring guess, and shrewd verification. In this respect, however, Euler is not unique; other mathematicians, great and small, used induction extensively in their work.

Yet Euler seems to me almost unique in one respect: he takes pains to present the relevant inductive evidence carefully, in detail, in good order. He presents it convincingly but honestly, as a genuine scientist should do. His presentation is "the candid exposition of the ideas that led him to those discoveries" and has a distinct charm. Naturally enough, as any other author, he tries to impress his readers, but, as a really good author, he tries to impress his readers only by such things as have genuinely impressed himself."

In his interesting article on Ars Expositionis: Euler as Writer and Teacher, G. L. Alexanderson (1983) describes an extraordinary ability of Euler as writer and teacher and says:

"In reading Euler's exposition, one cannot help but agree with Polya that from it we can learn "a great about mathematics, or the psychology of invention, or inductive reasoning [10, p. 99]." His techniques as well as his results are a bountiful source of ideas for modern researchers. ... he seems to be talking to the reader, explaining, something apologizing for the lack of rigor, but always giving insights into the process of discovery [6], [10, p. 17-21]." Reference [10] is the 1954 book of George Polya as stated before.

Unfortunately during his second St. Petersburg period from 1766 to 1783, Euler suffered from major health problems, and family disasters. In 1771, his house was badly burnt down in a fire of that year, and he lost almost all of his household properties including his library, but most of his books and research manuscripts were fortunately saved. However, his house was completely rebuilt with preservation of its original form and beauty. In addition, he became almost blind due to an unsuccessful surgery to remove cataract in his left eye in 1766. In 1773, Euler's first wife Catharina died, and three years later in 1766 he remarried to his first wife's half-sister, Salome Gsell. His two daughters died in 1780 and 1781. Fortunately, he had a phenomenal memory, prodigious powers of mental calculation, and his incredible ability to store long mathematical results and formulas in his memory for later dictation. Amazingly, Euler completed almost half of his works during his second 18-year stay in St. Petersburg. He continued his research on optics, algebra, lunar and planetary motion. His two-volume book on *Elements of Algebra* became the most successful mathematics textbook since *Euclid's Elements*. In these efforts, Euler received considerable help from his sons, Johann and Christopher, his student, N. I. Fuss and the academician, A. J. Lexell (1740-1784) who occupied Euler's position at the St. Petersburg Academy after Euler's death in 1783.

This short biographical sketch of Leonhard Euler would perhaps be incomplete without mention of his personal characteristics. From an early age, Leonhard was extremely conscious and intensely proud of his religious and Swiss cultural heritage. Throughout his life, he remained a Swiss citizen and maintained his family values and religious faith which he inherited from his parents who wished him to become a priest in his village church in Switzerland. He and his first wife Catharina raised their five children and then some of their grandchildren with love and care. He was always very cordial with his relatives, friends, students, and colleagues. Euler loved music and musical arts. He devoted all of his leisure time to music and musical theory. His love of music was directly linked to his mathematical investigations of the theory of music. His 1739 original treatise on music was a real testimony of his extensive knowledge and active involvement in music research. Among other subjects, his celebrated *Letters to a German Princess* also contained considerable discussion on music and analysis of musical sound.

Throughout his whole life, he was actively involved in numerous scientific correspondences in a friendly and respectful manner with many famous and contemporary mathematical scientists all over Europe. Euler's extensive scientific correspondence with notable mathematicians and scientists including Johann Bernoulli, Daniel Bernoulli, Christian Goldback, J. N. Delisle, P.L.M. Maupertuis, Alexis-Claude Clairaut (1713-1765), Jean d'Alembert, J. L. Lagrange, and Jean-Henri Lambert (1728-1777), had been his major sources of exchange of new ideas and discoveries which helped him to solve new problems and discover new results. It was also a real testimony of fruitful interaction between mathematical scientists through exchange of open questions, unsolved problems, conjectures, critical comments, praise and pleasure, and above all, an almost inexhaustible source of fresh creative ideas. After careful reading and analysis of these abundant correspondences, one cannot and but be simply impressed by his philanthropic nature, and by his sincere respect for every correspondent regardless of academic title, authority, or social stature. He not only showed respect for the point of view of an appropriate opponent, he always made a strong defense of mathematical or scientific truth. It is also appropriate to celebrate Euler's belief of mathematical and scientific discovery or truth by citing a famous quotation of S. Chandrasekhar (1910-1995) at the conclusion of his Nobel Lecture in December 8, 1983: "The simple is the seal of the true. And beauty is the splendor of truth." In general, Euler's character was that of a kind and gentle man. Along with spectacular successes of his many results and discoveries, some of Euler's work suffered from severe criticisms due to lack of rigor and clarity. However, he was never disappointed or discouraged by criticisms from others, indeed, he took them as a source of challenge for new discoveries. Even though he was one of the greatest mathematical scientists of his time and the recipient of many awards and honors, Euler retained his modesty and humility.

In order to illustrate his gracious praise and recognition of works of others, it may be appropriate to give a couple of examples. It was Euler who first discovered the calculus of variations in 1744. During 1760-1761, J. L. Lagrange developed a more general mathematical formulation of calculus of variations and sent a letter to Euler describing his general treatment of Euler's calculus of variations. After receiving Lagrange's correspondence, Euler immediately recognized superiority of Lagrange's work which led to joint recognition of the Euler-Lagrange equation associated with calculus of variations. While he was in Berlin, Euler received the two-volume research work of an Italian mathematician, C. G. Fagnano (1682-1766), entitled *Produzioni Matematiche*, published in 1750, for his review. Among other ideas and results dealing with elliptic integrals and their applications to the problem of rectification of an arc of an ellipse, hyperbola, lemniscate and cycloid, a new treatment of the duplication formula for the arc length of the famous lemniscate curve, $r^2 = a^2 \cos 2\theta$ in polar coordinates contained in these volumes. Fagnano was so proud of his work that he left instructions to inscribe a lemniscate on his grave. Impressed with the original work of Fagnano, Euler gave a rave review of it. Stimulated by Fagnano's work, Euler published a series of papers which laid the foundation of a new area of algebraic functions and their integrals including the remarkable addition and multiplication theorems for elliptic integrals. Motivated by applications, Euler's investigation of elliptic functions also began with his study of *elastica* which is the shape of a curve described by a thin elastic rod compressed at the ends. In 1757, Euler proved the famous addition theorems for elliptic integrals which led to the foundation of a subject of the theory of elliptic functions. It was Euler who first solved the problem of simple pendulum with finite amplitude in terms of elliptic functions. In one of his books on *Number Theory: An Approach Through History from Hammurapi to Legendre*, published in 1984, André Weil (1906-1998), one of the great mathematicians of the twentieth century, made a delightful comment on Euler's personality:

"With characteristic generosity Euler never ceased to acknowledge his indebtedness to Fagnano; but surely name but Euler would have seen in Fagnano's isolated results the germ of a new branch of analysis."

In his work, André Weil (1983) made another interesting remark on Euler's splendid personality and said that Euler was always open and receptive to new ideas and suggestions. In the words of Weil:

"...what at first is striking about Euler is his extraordinary quickness in catching hold of any suggestion, wherever it came from... There is not one of these suggestions which in Euler's hand has not become the point of departure of an impressive series of researches.... Another thing, not less striking, is that Euler never abandons a research topic, once it has excited his curiosity; on the contrary, he returns to it, relentlessly, in order to deepen and broaden it on each revisit. Even if all problems related to such a topic seem to be resolved, he never ceases until the end of his life to find proofs that are "more natural", "simpler", "more direct"."

As mentioned earlier, Euler translated the outstanding 150-page book on *New Principles of Gunnery* by Benjamin Robins from English to German. His German translation with large and extensive mathematical commentaries became almost over 700-page long book — a book of monumental work which was intended to include all mathematical and experimental knowledge of ballistics research at the time. On one hand, Euler made many positive criticisms and endorsement of Robins' new and brilliant research in ballistics, and on the other hand, the joint work of Euler and Robins revolutionized the ballistics research of the 18th and 19th centuries. In view of their notable joint work, both Euler and Robins can be regarded as the founding fathers of modern ballistics.

In the above and other instances, Euler actively participated in many historical debates on priority and superiority of scientific discoveries. He always graciously recognized and praised the work of others in many different ways.

Sturdily built, broad shouldered and with light-colored bright eyes, Euler was a life-long Swiss citizen and had enjoyed his excellent general health throughout his life except for his eye problems. As a human being, he loved his profession, family and friends, and faithfully fulfilled all of his duties and obligations to society. Even when he was completely blind, his age showed no strain in his instantaneous photographic memory, intellectual power and imagination, and even his extraordinary ability of carrying out long and complex calculations. All of his students and colleagues respected him professionally and admired him personally. Even his blindness did not prevent him from doing his research and creative activities. He continued his work with the help of his two sons, Johann and Christopher, his research assistant, Fuss and the academician, Lexell. In early September of 1783, Euler began to suffer from dizziness which presaged his death. Subsequently, Euler died suddenly in St. Petersburg on September 18, 1783 at the age of 76 as a result of a stroke while playing with his grandson.

It is appropriate to mention that N. I. Fuss delivered the remarkable Eulogy in Memory of his teacher, Leonhard Euler at the St. Petersburg Academy meeting on October 23, 1783. This is an authoritative and thoughtful eulogy, originally presented in French, describing Euler's life, work, professional career and charming personality. Almost in the end of the eulogy, Fuss made the following memorable statement:

"Such are the works of Euler, such the feats worthy of perpetual remembrance. Posterity will join his name to those of the great Galileo, Leibniz, Newton, and almost all who have honored humankind through their intellect; his name will be remembered when those of so many others who owe their fleeting moment of renown to the vanity of our age are gone to everlasting oblivion.

There have been few scientists who have written so much as Euler, but there are none to compare with him in number and variety of mathematical discoveries."

With deep respect, gratitude and admiration, Fuss concluded his eulogy by adding the following magnificent statement:

"My dear sirs, any attempt of mine to portray to you that delightful scene of domestic bliss would be in vain. Many of you were, like me, eyewitnesses! Above all those of you assured of fame through having had such a teacher! There are five such former students here; what scientist can boast that he united in a single collective so many of his students? Let us express before all our eternal and most fervent gratitude and so demonstrate that our incomparable teacher evokes astonishment as much for his rare goodness as for his extraordinary intellectual power. Friends! Academicians! Mourn him with the sciences, which have never suffered such a loss, with his family of which he was adornment and support! My tears and your flow together; his benefactions, especially to me, will to the end of my life remain ineradicably with me."

In reading Fuss's authoritative eulogy, we can learn a great deal about many aspects of Euler's life, career and great discoveries which will remain an unlimited source of ideas for the development of modern mathematics and science.

André Weil also described Euler's legacy in his own words in his 1984 book as follows:

"No mathematician ever attained such a position of undisputed leadership in all branches of mathematics, pure and applied, as Euler did for the best part of the eighteenth century."

Another celebrated mathematical scientist of the twentieth century, John von Neumann described Leonhard Euler as "the greatest virtuoso of the period," for his invaluable contributions and their extraordinary impact on mathematics, science and society for over three centuries.

In mathematics, the eighteenth century can fairly be labeled as the era of Euler. However, his remarkable influence on the development of mathematical sciences was not restricted to that period only. His extensive research, lucid writings, tremendous energy, and his mathematical insights helped explore not only the mathematics and science of his time, but the life of his professional colleagues, their new opportunities and aspirations. The work of many outstanding nineteenth- and twentieth-century mathematicians arose directly from his extraordinary influence. It is hoped that this excursion into the wonderful life and career of Euler would provide the reader with adequate motivation to explore mathematics and science further in the twenty first century. There is no doubt that in a hundred years' time mathematical scientists will again celebrate the four hundredth anniversary of Euler's birth with the same delight and enthusiasm as we have commemorated the tercentenary in 2007. It would be very fascinating to know which areas of Euler's work will be resonating with the mathematical and scientific communities by then.

Euler's simple life and brilliant career was totally dedicated to the pursuit of fundamental mathematical and scientific discovery, and dissemination of new information and knowledge. His unique image in the research and teaching of contemporary mathematical and physical sciences is still extremely predominant. There is no doubt at all about his significant and ever-lasting impact on modern mathematics, modern science and culture. In many important ways, Euler made a significant and permanent contributions to the welfare of human race. He will be remembered forever not only for his great and universal achievement, but also for his unique contributions to the welfare of humankind.

Chapter 3

Euler's Contributions to Number Theory and Algebra

> "No mathematician ever attained such a position of undisputed leadership in all branches of mathematics, pure and applied, as Euler did for the best part of the eighteenth century."

André Weil

> "All celebrated mathematicians now alive are his disciples: there is no one who ... is not guided and sustained by the genius of Euler."

Marquis de Condorcet

3.1 Introduction

In the century before Euler, Pierre de Fermat spent his whole life in an extensive investigation on the theory of numbers and discovered a wide variety of ideas and results in number theory and formulated the fundamental principle of geometrical optics. Probably, Euler received considerable inspiration for his research in number theory from his study of Fermat's work. Euler made an extensive research correspondence with his friend and colleague, Christian Goldback on various problems in number theory. Undoubtedly, Euler also received many new ideas and results of number theory from his correspondence with Goldback.

3.2 Euler's Phi Function and Cryptography

One of Euler's fundamental contributions to number theory was a generalization of *Fermat's Little Theorem* which states that if p is prime and a

is prime to p, that is, $(a, p) = 1$, then $a^{p-1} - 1$ is divisible by p. In other words,

$$a^{p-1} \equiv 1 \pmod{p}. \tag{3.2.1}$$

For example, if $p = 7$ and $a = 2$, then $2^6 - 1 = 63$ is divisible by 7. If $p = 5$ and $a = 2, 3, 4$, then $2^4 \equiv 1 \pmod 5$, $3^4 \equiv 1 \pmod 5$, and $4^4 \equiv 1 \pmod 5$.

In 1760, Euler made a remarkable generalization by introducing a new function $\phi(n)$, known as the *Euler phi* (or *totient*) *function* which is defined as the number of positive integers $\phi(n) = r$ less than n and relatively prime to n, that is, $1 \le r < n$ and $(r, n) = 1$. Euler's theorem states that, if $(a, n) = 1$, then $a^{\phi(n)} - 1$ is divisible by n, that is,

$$a^{\phi(n)} \equiv 1 \pmod{n}. \tag{3.2.2}$$

It follows from the definition of $\phi(n)$ that $\phi(1) = 1$, and if p is prime, then $\phi(p) = p - 1$. Euler's ϕ function is a *multiplicative function*, that is, $\phi(n \cdot m) = \phi(n) \cdot \phi(m)$ for all $(n, m) \equiv 1$ holds. According to the fundamental theorem of arithmetic, every natural number $n(> 1)$ has a unique factorization in terms of prime numbers, that is,

$$n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m} = \prod_{r=1}^{m} p_r^{k_r}, \tag{3.2.3}$$

where p_m are the m distinct prime factors of n and k_m are positive integers. For example, $n = 30$, then $30 = 2 \times 3 \times 5$. If $n = 72$, then $72 = 8 \times 9 = 2^3 \cdot 3^2$.

If p is a prime, and k is a positive integer, then $\phi(p^k) = p^k - p^{k-1} = p^k \left(1 - \frac{1}{p}\right)$. So, in general, if n is of the form (3.2.3), then

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right). \tag{3.2.4}$$

This is a general formula for Euler's phi function in terms of prime factors. For example, $\phi(30) = 30 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) = 8$ and

$$\phi(72) = \phi\left(2^3 \cdot 3^2\right) = 72 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) = 24. \tag{3.2.5}$$

Using the prime factorization of n, it is possible to find another formula for $\phi(n)$, which follows directly from (3.2.4), in the form

$$\phi(n) = p_1^{k_1 - 1}(p_1 - 1)\, p_2^{k_2 - 1}(p_2 - 1) \cdots p_m^{k_m - 1}(p_m - 1). \tag{3.2.6}$$

For example, $\phi(72) = \phi\left(2^3 \cdot 3^2\right) = (2-1)2^{3-1} \cdot (3-1)3^{2-1} = 24$.

On the other hand, for a composite number n, we obtain totally new and often unexpected results. For example, for $n = 10$, $\phi(n = 10) = 4$ as

$r = 1, 3, 7, 9$. Then $a^4 - 1$ is divisible by 10. In other words, the fourth power of any number not containing the factors 2 or 5 has 1 as the last digit so that $3^4 = 81$, $7^4 = 2401$, $9^4 = 6561$, $13^4 = 28,561$, and so on.

The Euler phi-function has new and modern applications to *cryptography* (or *the science of secret codes*) which deals with safeguarding and sending secret messages securely. In simple words, a message M is transmitted by a sender to a receiver in digital encrypted form. The sender encodes the message M into E so that the receiver can decode E back into the original message M. Mathematically, the encoded (or encrypted) message $E = f(M)$ can be represented by the number

$$E \equiv M^s \qquad (\text{mod } r), \tag{3.2.7}$$

where s is called an *encoded* (*encrypted*) *exponent*. So, it is relatively easy to obtain E for any large exponent s. The number r and s form the *public key* as they are known to the general public. The exponent s is chosen so that s and $\phi(r)$ are coprime, that is, $(s, \phi(r)) = 1$. The major problem is to obtain M from E, r and s (or to simply invert M from E) which is also known as the *decoding* (or *decrypting*) *problem* and it is a very difficult problem. In 1976, Whitfield Diffie and Martin Hellman published a landmark work describing the *public-key encryption* system with the aid of *one-way trap door functions*. Based on the idea of the trap door function and the Euler phi function, three mathematicians, Ronald Rivest, Adi Shamir and Leonhard Adelman at the M. I. T. discovered a new and effective method, known as the *RSA encryspton method* which provided the solution of the cryptographic problem.

In order to solve the decoding problem, the receiver has to use a *private key* formed by the exponent t and r so that $ts \equiv 1 \pmod{\phi(r)}$, or $ts = 1 + k\,\phi(r)$ for some integer k. Thus, decoding the message with exponent t can be done as follows:

$$E^t \equiv M^{st} \equiv M^{1+k\,\phi(r)} \qquad (\text{mod } r) = M, \tag{3.2.8}$$

since $M^{\phi(r)} \equiv (\text{mod } r)$.

To encode the message M, two distinct and very large prime numbers p and q are chosen with $r = pq$ so that $\phi(r) = \phi(pq) = \phi(p)\phi(q) = (p-1)(q-1)$, and the encoded message is given by (3.2.7). The computation of the decoding exponent t requires values p and q. In general, M, s, r and t are very large numbers so that computations seem to be almost a formidable task. However, there are effective methods of computing these numbers (see Silverman, J.H. (2006)). Clearly, the Euler phi function played a major and effective role in secure transmission of secret codes and ciphers.

Considerable recent studies have been made about security offered by the RSA. The application of cryptography to computer data security has extensively been investigated in recent years. Probably, Euler never dreamed of such applications of his phi function to cryptography and modern public-key encryption systems.

3.3 Euler's Other Work on Number Theory

In a letter to Euler in 1742, Goldback conjectured that every even number greater than 2 is the sum of two prime numbers. For example, $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, $12 = 5 + 7$. Computer searches have numerically confirmed to Goldback conjecture for all even numbers up to 10^{10}. Euler replied that he believed, but could not prove it, and formulated the stronger version that every even integer (> 4) is the sum of two primes. In 1937, a Russian mathematician Ivan M. Vinogradov (1891-1983) proved that any sufficiently large odd integer is the sum of at most three prime numbers. However, this conjecture has not yet been proved or disproved. This is one of the unsolved problem in number theory.

In 1770, both Euler and Lagrange proved the *Four-Square Problem*, that is, every positive integer, n is the sum of at most four square integers:

$$n = x^2 + y^2 + z^2 + w^2. \tag{3.3.1}$$

In 1783, Euler gave a proof of *Wilson's* (1741-1793) *theorem* that if and only if p is prime, then

$$(p-1)! + 1 \equiv 0 \pmod{p}. \tag{3.3.2}$$

For example, $p = 2, 3, 5, \cdots$, are primes because (3.3.2) holds. John Wilson, a senior Wrangler in mathematics of the University of Cambridge, conjectured the result (3.3.2), but could not prove it. His teacher Edward Waring (1734-1798) of the University of Cambridge published it under the name of Wilson. The Fermat theorem (3.2.1) can be used to prove Wilson's theorem (3.3.2). In fact, (3.2.1) holds for $a = 1, 2, 3, \cdots, p-1$. According to the fundamental theorem of algebra, these $p-1$ roots must be all the roots of (3.2.1) so that

$$a^{p-1} - 1 \equiv (a-1)\cdot(a-2)\cdots(a-p+1) \pmod{p}. \tag{3.3.3}$$

Putting $a = p$ gives

$$p^{p-1} - 1 \equiv (p-1)(p-2)\cdots 1 = (p-1)! \pmod{p}$$

and $p^{p-1} \equiv 0 \pmod{p}$. Thus, Wilson's theorem (3.3.2) follows.

Fermat had conjectured that all numbers of the form

$$F_n = 2^{2^n} + 1, \qquad n = 0, 1, 2, 3, 4, \cdots \tag{3.3.4}$$

are primes. Fermat's conjecture is true for $n = 0, 1, 2, 3, 4$ as it can easily be verified and so, only 5 Fermat primes are known today. In 1732, Euler proved a remarkable result, that is, the fifth Fermat number, F_5 is *not* prime, but composite, as it is divisible by 641 because

$$F_5 = 2^{2^5} + 1 = 4294967297 = 641 \times 6700417. \tag{3.3.5}$$

The Fermat numbers F_6, F_7, F_8 are composite and some of their factors are now known, and at least 200 Fermat numbers are now known to be composite including F_n, $n = 2478782$ discovered by John Cosgrave and his associates at the St. Patrick College, Dublin in 2003. It is important to note that the Fermat primes are of special interest in plane geometry. Gauss proved one of the most remarkable results which relates the Fermat primes with the sides of a regular polygon. More precisely, if F_n is a prime p, then a regular polygon of p sides can be inscribed in a circle by Euclidean methods.

Marin Mersenne had conjectured that there are only finitely many primes of the form $M_p = (2^p - 1)$, when p is a prime. These are called *Mersenne primes*. Only 44 Mersenne primes are known today. If $p = 11$, $2^{11} - 1 = 23 \times 89$. As of 2004, the largest known Mersenne prime is $2^p - 1$ when $p = 24036583$, only the forty-first Mersenne prime number. It was found by the *Great Internet Mersenne Prime Search (GIMPS) project.* To each Mersenne prime, $M_p = (2^p - 1)$, there is an associated perfect number, $P = M_p \cdot 2^{p-1}$. A positive integer n is called a *perfect number*, if the sum of its proper divisors (other than the number itself) $s(n) = n$. Hence, $\sigma(n) = s(n) + n = 2n$. If $\sigma(n) < 2n$ or $\sigma(n) > 2n$, then n is called *deficient* or *abundant* respectively. Historically, the concepts of perfect number, deficient number and abundant number were discovered by Pythagoreans. One of the Pythagoreans used the Euclid method to generate perfect numbers which remains today the only method known to generate such numbers. If the sum of the numbers 1, 2, 4, $\cdots$, 2^{n-1} is prime, then the sum multiplied by the last term is perfect. For example, $1 + 2 = 3$, which is prime. Then, 3 is multiplied by the last term, 2 gives 6 which is a perfect number because the sum of its divisors 1, 2, 3 is 6. Similarly, $1 + 2 + 4 = 7$ which is prime so that $4 \times 7 = 28$ which is the next perfect number. Then $1+2+4+8+16 = 31$ which is also prime so that $31 \times 16 = 496$ is a perfect number. The next perfect number is $8128 = (1+2+4+8+16+32+64) \times 64 = 127 \times 64$. This method led Nichomachus to find the first four perfect numbers. As of today, there are exactly 41 perfect numbers known.

In 1732, Euler discovered the 19-digit *eighth perfect number* $P = 2^{31-1}(2^{31}-1)$ when $p = 31$. In the book 1X of his *Elements*, Euclid proved in about 350-300 BC that if $2^p - 1$ is prime, the number

$$P = 2^{p-1}(2^p - 1) \tag{3.3.6}$$

is *perfect*. Two thousand years later, Euler showed that every *even perfect number* is of this type. Indeed, there are *no* known odd perfect numbers. It is conjectured that all perfect numbers are even. Although this has not yet been proved, some evidence has been found in favor of this conjecture. If an odd perfect number exists, it is known that it must be greater than 10^{300} and have at least 9 distinct prime factors. On the other hand, every perfect number is a sum of consecutive odd cubes. For example,

$$28 = 1^3 + 3^3, \qquad 496 = 1^3 + 3^3 + 5^3 + 7^3. \tag{3.3.7}$$

If n is perfect, then the sum of the reciprocals of all divisors of n is always equal to 2. For example, 6 is perfect and has divisors 1, 2, 3, 6, and hence,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2. \tag{3.3.8}$$

The number 28 is also perfect with divisors 1, 2, 4, 7, 14, 28, and share the same property.

Around 250 A.D., an ancient Greek mathematician, Diophantus of Alexandria sought *solutions in integers* or *rational numbers* of the so called *Diophantine equations* of the form

$$x^n + y^n = z^n, \qquad n \geq 3. \tag{3.3.9}$$

This is universally known as the *Fermat Last Theorem* (*FLT*). Fermat gave a proof of (3.3.9) for $n = 4$. It was Euler who proved the FLT for $n = 3$. The significance of Euler's proof was that it made use of number theory in complex numbers, by introducing $\mathbf{Q}[\sqrt{-3}]$, where $\mathbf{Q}$ is the field of rational numbers with the property of unique factorization of non-zero numbers as products of prime numbers. In 1851, a German mathematician, Ernst Eduard Kummer (1810-1893) made the major breakthrough in the introduction of modern algebraic number theory and its successful applications to FLT in many cases. He showed that the FLT is true for the so called irregular primes. The true breakthrough was made by Leuis Joel Mordell (1888-1972) who conjectured that (3.3.9) has at most a finite number of solutions. In 1983, Gerd Faltings (1954-) proved that the Mordell conjecture for Diophantine equations is true. In spite of numerous attempts made by many world's famous mathematicians, it was not until 1994 that Andrew Wiles proved the FLT by proving the 1955 Shimura-Taniyama-Weil geometric conjecture about elliptic curves. Finally, Andrew Wiles (1953-) successfully solved the Fermat Last Theorem in 1995.

Euler formulated a remarkable conjecture that

$$x_1^n + x_2^n + \cdots + x_k^n = z^n \tag{3.3.10}$$

has nontrivial integer solutions if and only if $k \geq n$. For $n = 3$ and $k = 2$, Euler's conjecture corresponds to the proved Fermat Little Theorem. For $n = 3$ and $k = 3$, (3.3.10) implies that the sum of three cubes can be another cubic. Euler's conjecture remained valid for over two centuries. In 1966, Lander and Parkin used the CDC 6600 computer to discover a counter example for $n = 5$, that is,

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5. \tag{3.3.11}$$

Euler also discovered a remarkable property of the quadratic expression with integral n,

$$f(n) = n^2 + n + 41. \tag{3.3.12}$$

For $n = 0$ to 39, the exact value of $f(n)$ is a prime number. For example, $f(0) = 41$, $f(1) = 43$, $f(2) = 47$ which are primes. Indeed, the expression (3.3.12) produces a large number of primes, but $f(40) = 41^2 = 1681$ which is *not* a prime. Of the first 40 million values the proportion of primes is about one in three — far greater than any other quadratic formula. In fact, Euler's quadratic expression (3.3.12) seems to be unusual in its production of prime numbers. What happens when n is replaced by a real or even by a complex number? Such a questions is investigated in a new branch of pure mathematics, known as analytic number theory. When n is replaced by a real number x, there are complex roots of the Euler quadratic equation $f(x) = 0$ so that

$$x = -\frac{1}{2} \pm i \cdot \frac{1}{2}\sqrt{163}. \tag{3.3.13}$$

What is so special about $\sqrt{163}$? In fact, it is the largest value of d for which the number system $a + ib\sqrt{d}$ allows unique factorization. With each number system derived from some value of d, Gauss identified a certain natural number $h(d)$ called the *class number* of that system. He also described an extensive computation of class numbers, and observed that, for each class number k, there exists a largest value of d for which $h(d) = k$. The largest d with $h(d) = 1$ was $d = 163$. In Gauss' time, nine values of $d = 1, 2, 3, 7, 11, 19, 43, 67$ and 163 were known for which the system of numbers $a + b\sqrt{-d}$ has the unique factorization with the largest $d = 163$. This explained why number 163 was so special in Euler's quadratic equation. On the other hand, the largest d for which $h(d) = 2$ seemed to be $d = 427$, and the largest d with $h(d) = 3$ was 907. Gauss was neither able to confirm that any of these values really was the largest, nor prove that there always was a largest d. However, Gauss conjectured that this would be the case. In 1952, this class number problem was solved for the case $h(163) = 1$ by a retired Swiss mathematician, Kurt Heegner. However, nobody believed his proof because his paper was hard to understand. Another fifteen years later in 1967, Harold Stark (1939-) of the Massachusetts Institute of Technology and Alan Baker (1939-) of the University of Cambridge provided independently different proofs to establish that there is no tenth d.

In his famous 1750 paper with a fascinating title, "*De Numeris Amicabilibus*", Euler began his extensive study of amicable numbers. A pair of numbers (m, n) is called a pair of *amicable numbers* if the sum of the *proper* divisors of m (except m) is equal to n and vice-versa. The *smallest amicable pair* is $(220, 284)$ because the sum of the proper divisor of 220 is $1+2+4+5+10+11+20+22+44+55+110 = 284$, and the sum of the proper divisors of 284 is $1 + 2 + 4 + 71 + 142 = 220$. Only three amicable pairs were known before Euler. In 1747, Euler published a short paper with a list of 30 amicable pairs using the method of Descartes and Fermat. In his 1750 paper, Euler introduced a new *number theoretic function*, known as the *Euler sigma function*, $\sigma(m)$ which is defined as the sum of all divisors of a given number m and developed a new method of finding amicable pairs. For example, $m = 21$, $\sigma(21) = 1 + 3 + 7 + 21 = 32$.

There are immediate characterizations of prime numbers, that is, p is prime, if and only if $\sigma(p) = p + 1$. More generally, if p and q are different primes, then

$$\sigma(pq) = \sigma(p)\sigma(q). \tag{3.3.14}$$

For example $\sigma(21) = \sigma(3 \cdot 7) = \sigma(3)\sigma(7) = 4.8 = 32$. Euler proved a more general result, that is, the multiplicative result holds not just for different prime numbers, but for any whole number whose greatest common divisor is 1. More explicitly, Euler proved the following theorem: If $gcd(a, b) = 1$, then

$$\sigma(ab) = \sigma(a)\sigma(b). \tag{3.3.15}$$

For example, to determine the sum of all divisors of 585, we write it into relatively prime factors and use the above theorem. Thus,

$$\sigma(585) = \sigma(5 \cdot 9 \cdot 13) = \sigma(5)\sigma(9)\sigma(13) = 6 \cdot 13 \cdot 14 = 1092. \qquad (3.3.16)$$

Euler used his sigma function to reformulate the definition of amicable number pairs by noting that the sum of the *proper divisors* of a whole number of n is $\sigma(n) - n$. Consequently, m and n are amicable pairs if and only if $\sigma(m) - m = n$ and $\sigma(n) - n = m$. This leads to Euler's famous definition that m and n are amicable if and only if

$$\sigma(m) = m + n = \sigma(n). \qquad (3.3.17)$$

Euler used this elegant characterization as his method of testing in the world of amicable numbers.

In 1737, Euler proved that the number of primes is infinite by showing that the sum of their reciprocals diverges, that is

$$\sum_{n=\text{prime}} \frac{1}{n} = \infty. \qquad (3.3.18)$$

Without any question about convergence, in 1748 he proved that

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} = \prod_{p=2}^{\infty} (1 - p^{-x})^{-1}, \quad p \text{ is prime} \qquad (3.3.19)$$

where $\zeta(x)$ is the *Euler zeta function*. He also recognized at that time that there was a connection between the zeta function and the distribution of prime numbers.

This result reduces to (3.3.18) in the limit at $x \to 1$. It is now known that the number of primes is infinite. We next discuss another major question dealing with the number of primes less than or equal to x which is denoted by $\pi(x)$. For example, $\pi(2) = 1$, $\pi(3) = 2$, $\pi(10) = 4$, $\pi(17) = 7$ and so on. This leads to the equivalent statement of the Euclid theorem as

$$\lim_{x \to \infty} \pi(x) = \infty. \qquad (3.3.20)$$

This function $\pi(x)$ has been the subject of intense research for the last several hundred years. The precise nature of $\pi(x)$ as $x \to \infty$ has become known as the celebrated *Prime Number Theorem*:

$$\lim_{x \to \infty} \frac{\pi(x)}{(x/\ln x)} = 1. \qquad (3.3.21)$$

Or, equivalently, as $x \to \infty$

$$\pi(x) \sim \frac{x}{\ln x}, \quad \text{or} \quad \pi(x) \sim L(x) = \frac{x}{\ln x - a(x)},$$

or

$$\pi(x) \sim \mathrm{Li}\,(x) = \int_2^\infty \frac{du}{\ln u}. \tag{3.3.22}$$

Among all asymptotic approximations for $\pi(x)$ for large x obtained by many great mathematicians, the function $\mathrm{Li}\,(x)$ provides a much closer asymptotic approximation to $\pi(x)$ than does $(x/\ln x)$ or $L(x)$. Euler, Legendre and Gauss suggested the Prime Number Theorem (3.3.21) without a conclusive proof. It is natural to ask about the distribution of primes, that is, how the primes are distributed on the positive real axis. It follows from the available data that over a long intervals the density of primes tends to decrease as one approaches larger and larger integers.

However, Euler asserted the following functional equation for real x

$$\zeta(1-x) = 2(2\pi)^{-x} \cos\left(\frac{\pi x}{2}\right) \Gamma(x)\,\zeta(x). \tag{3.3.23}$$

In 1859, Riemann introduced the zeta function $\zeta(z)$ for complex $z = x + iy$ and proved (3.3.23) for complex $z = x + iy$ and then used $\zeta(z)$ to prove the Prime Number Theorem. He generalized the Euler result (3.3.19) for complex z in the form

$$\frac{1}{\zeta(z)} = \prod_{p=2}^{\infty} \left(1 - \frac{1}{p^z}\right). \tag{3.3.24}$$

Thus, the Riemann zeta function is closely associated with the distribution of prime numbers. Indeed, the asymptotic distribution of primes is related to the singularity of the zeta function. It can be shown that, for large x,

$$\pi(x) \sim \int_2^x \frac{du}{\ln u} = \mathrm{Li}(x). \tag{3.3.25}$$

This gives the asymptotic distribution of the prime numbers which first conjectured in 1791 by Gauss, and then finally and independently proved by Jacques Hadamard (1865-1963) and Charles-Jean de La Vallée Poussin (1866-1962) in 1896. However, it may be appropriate to mention some remarkable progress made by the Russian mathematician P. L. Chebyshev (1821-1894) toward a proof of the prime number theorem in 1852 and 1854 based on the Euler zeta function $\zeta(x)$ for real x. In fact, he proved that

$$0.92129 \le \lim_{x\to\infty} \inf \frac{\pi(x)}{(x/\ln x)} \le 1 \le \lim_{x\to\infty} \sup \frac{\pi(x)}{(x/\ln x)} \le 1.10555. \tag{3.3.26}$$

About fifty years after the Hadamard-de la Vallée Poussin proof, Paul Erdös (1913-1996) and Atle Selberg (1917-2007) gave independently an *elementary proof* of the prime number theorem in 1948 without any knowledge of complex function theory. But their proofs are still very long and complicated. It was Norbert Wiener (1894-1964) who derived the prime number theorem from the Wiener-Ikehara Tauberian theorem. This is perhaps a more transparent proof from the analytical point of view.

Riemann pointed out that further investigation of the zeta function requires information about the complex zeros of $\zeta(z)$, and then conjectured that all non-trivial zeros of the zeta function $\zeta(z)$ lie on the line $z = \frac{1}{2} \pm iy$. This is universally known as the *Riemann Hypothesis* which has not yet been proved or disproved. If the Riemann Hypothesis is true, the known estimate $\pi(x) = \mathrm{Li}(x) + O[x \exp(-c\sqrt{\ln x})]$ would become $\pi(x) = \mathrm{Li}(x) + O(\sqrt{x} \ln x)$. In other words, the validity of the Riemann Hypothesis is equivalent to the statement that $|\pi(x) - \mathrm{Li}(x)| \leq a\sqrt{x}\, \ln x$ for some constant a. According to the 1974 Fields Prize Winner mathematician, Enrico Bombieri (1940-), it seems very hard to make further improvement of the above estimate because of Littlewood's theorem that the degree of oscillation $\pi(x) - \mathrm{Li}(x)$ is asymptotically of the order $\mathrm{Li}\,(\sqrt{x}) \ln \ln \ln x$. It is worth to quote Bombieri's statement: "The failure of the Riemann Hypothesis would create havoc in the distribution of prime numbers".

In 1896, Hadamard applied the theory of entire functions of a complex variable to prove the Prime Number Theorem based on crucial fact that $\zeta(z) \neq 0$ for $x = 1$. On the other hand, in 1896, Vallée Poussin used some properties of the zeta function, and finally proved the Prime Number Theorem. We close this section, by quoting A. E. Ingham's (1995) words from his treatise *The Distribution of Prime Numbers*:

"The solution (of the Prime Number Theorem) just outlined (that of de la Vallée Poussin and Hadamard) may be held to be unsatisfactory in that it introduces ideas very remote from the original problem, and it is natural to ask for a proof of the Prime Number Theorem not depending on the theory of functions of a complex variable. To this we must reply that at present no such proof is known. We can indeed go further and say that it seems unlikely that a genuinely 'real variable' proof will be discovered, at any rate so long as the theory is founded on Euler's identity. For every known proof of the Prime Number Theorem is based on a certain property of the complex zeros of $\zeta(s)$, and this conversely is a simple consequence of the Prime Number Theorem itself. It seems clear therefore that this property must be used (explicitly or implicitly) in any proof based on $\zeta(s)$, and it is not easy to see how this is to be done if we take account only of real values of s."

3.4 Euler and Partitions of Numbers

During his stay in Berlin, a mathematician, Phillip Naude (1684-1747) of French origin, raised a number of mathematical questions in his letter to Euler in 1740. One of his question was: In how many ways can integer n be represented as a sum of integers? In response to this question, Euler discovered many new ideas, results and methods of partitions of numbers. He presented many elementary but remarkable results in his fundamental treatise on analysis, *Introductio in analysin infinitorum*.

The *partition function*, $p(n)$ is defined to be number of ways of writing a positive integer n as a sum of strictly positive integers. For example: $6 = 5+1 = 4+2 = 4+1+1 = 3+3 = 3+2+1 = 2+2+2 = 2+2+1+1 = 3+1+1+1 = 2+1+1+1+1 = 1+1+1+1+1+1$ so that $p(6) = 11$. Similarly, the partitions of 6 into odd parts are $5+1 = 3+1 = 3+1+1+1 = 1+1+1+1+1+1$ so that the number of partitions is 4. The numbers of partitions of 6 into distinct parts $(6, 5+1, 4+2, 3+2+1)$ is also 4. The number 6 has only *one* partition into distinct odd parts: $5+1$. This illustrates a simple idea of the concept of unrestricted and restricted partitions of an integer n. The restrictions may sometime be so stringent that $p(n)$ does not exist. For instance, 10 cannot be partitioned into three distinct odd parts.

Partitions have an inherent symmetry. The geometrical representation of a partition, known as *Ferrers graphs*, is given by dots which is introduced by a British mathematician, Norman M. Ferrers (1829-1903). In quantum mechanics, such geometrical representations are known as *Young's tableaux* which is introduced for the study of symmetric groups. They are also found to be useful for the investigation of the symmetries of many-electron systems.

Euler proved many important theorems in the theory of partitions. He considered a power series in the form

$$F(x) = \sum_{n=0}^{\infty} p(n)\, x^n, \tag{3.4.1}$$

where $F(x)$ is called the *generating function* of the partition function $p(n)$. Based upon this generating function, Euler formulated the analytical theory of partitions by proving a simple and remarkable result:

$$F(x) = \sum_{n=0}^{\infty} p(n)x^n = \prod_{m=1}^{\infty} \left(1 + x^m + x^{2m} + \cdots\right) = \prod_{m=1}^{\infty} (1-x^m)^{-1}, \tag{3.4.2}$$

where $p(0) = 1$. This is known as the *Euler Theorem* provided $|x| < 1$.

It also follows from (3.4.2) that the generating function $F_m(x)$ for the partition of n into integers the largest of which is m has the form:

$$F_m(x) = \frac{1}{\prod_{k=1}^{m}(1-x^k)} = \frac{1}{(1-x)(1-x^2)\cdots(1-x^m)}. \tag{3.4.3}$$

Similarly, the generating function for the partition of n into distinct integers is

$$F(x) = (1+x)\left(1+x^2\right)\left(1+x^3\right)\cdots. \tag{3.4.4}$$

This result can be rewritten as

$$F(x) = \frac{1-x^2}{1-x}\cdot\frac{1-x^4}{1-x^2}\cdot\frac{1-x^6}{1-x^3}\cdots \tag{3.4.5}$$

$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = \prod_{m=1}^{\infty}\left(1-x^{2m-1}\right)^{-1}. \tag{3.4.6}$$

The right-hand side of this result is obviously the generating function for the partition of n into odd integers. This also gives a remarkable result which says that the number of partitions of n into unequal parts is equal to the number of its partitions into odd parts.

Another beautiful result follows from Euler's Theorem, and it has the form

$$\frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^3)(1-x^5)\cdots} = 1+x+x^3+x^6+x^{10}+\cdots. \tag{3.4.7}$$

The powers of x in (3.4.7) are the familiar *triangular numbers*, $\triangle_n = 1+2+3+\cdots+n = \frac{1}{2}n(n+1)$ that can also be represented *geometrically* as the number of equidistant points in triangles of different sizes. These points form a triangular lattice as shown in Figure 3.1. Thus, the triangle numbers $\triangle_n$ are 0, 1, 3(1 + 2), 6(3 + 3), 10(6 + 4), 15(10 + 5), $\cdots$. Their first differences form a linear progression 1, 2, 4, 5, $\cdots$. As a generalization

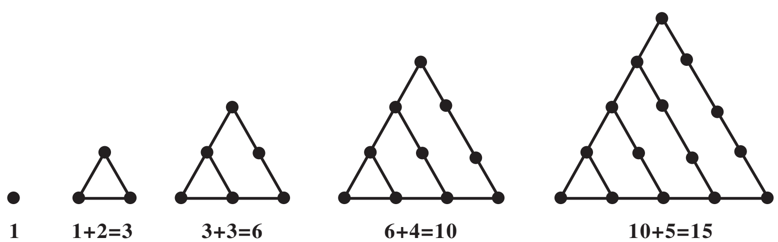


Fig. 3.1 The triangular numbers $\triangle_n = 1+2+3+\cdots+n = \frac{1}{2}n(n+1)$.

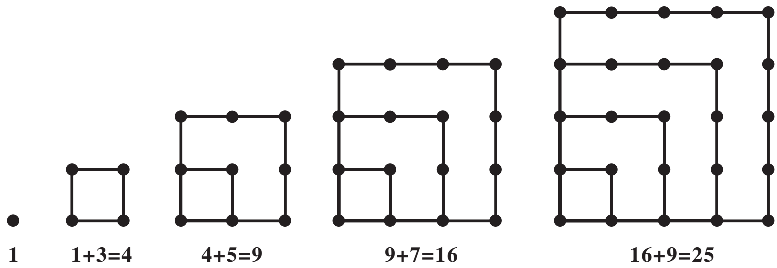


Fig. 3.2 The Square numbers $\square_n = (n+1)^2$.

of this idea, square numbers are defined by the number of points in square lattices of increasing size, that is 1, $4(1+3)$, $9(4+5)$, $16(9+7)$, $25(16+9)$, $\cdots$. In other words, the square numbers are $\square_n = (n+1)^2$ as shown in Figure 3.2. It is possible to generalize these numbers for n-gonal numbers (such as pentagonal and hexagonal numbers) $f_n(k)$ given by

$$f_n(k) = \frac{1}{2}(n-2)k^2 + \frac{1}{2}nk + 1, \tag{3.4.8}$$

when $n=3$, $f_3(k) = \frac{1}{2}(k+2)(k+1)$ (*triangular numbers* when $k = 0, 1, 2, 3, 4, 5, 6, \cdots$), $n=4$, $f_4(k) = (k+1)^2$ (*square numbers*), $n=5$, $f_5(k) = \frac{1}{2}(k+1)(3k+2)$ (*pentagonal numbers*); and $n=6$, $f_6(k) = (k+1)(2k+1)$ (*hexagonal numbers*).

The sum of the triangular numbers is

$$\sum_{n=1}^{n} \triangle_n = \frac{1}{2}\left(\sum_{n=1}^{n} n^2 + \sum_{n=1}^{n} n\right) = \frac{n(n+1)(n+2)}{6}. \tag{3.4.9}$$

The sum of the reciprocal of the triangular numbers is

$$\begin{aligned}\sum_{n=1}^{n} \frac{2}{n(n+1)} &= \frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \cdots \\ &= 2\left(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \cdots\right) \\ &= 2\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots\right] = 2. \end{aligned} \tag{3.4.10}$$

In his article, Nelson (2008) presents many classical results of elementary number theory involving triangular numbers $\triangle_n$. He illustrated relatively less known pattern which involves longer and longer sums of consecutive squares including

$$3^2 + 4^2 = 5^2,$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2,$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2.$$

An algebraic proof of the general identity

$$(4\triangle_n - n)^2 + \cdots + (4\triangle_n)^2 = (4\triangle_n + 1)^2 + \cdots + (4\triangle_n + n)^2 \quad (3.4.11)$$

is available in the literature. In fact, we write

$$\left[(4\triangle_n + 1)^2 - (4\triangle_n - 1)^2\right] + \cdots + \left[(4\triangle_n + n)^2 - (4\triangle_n - n)^2\right] = (4\triangle_n)^2 . \quad (3.4.12)$$

In 1750, Euler ingeniously proved that the partition function generated by the product $\prod_{m=1}^{\infty}(1 - x^m)$, the reciprocal of the generating function of $p(n)$, has a surprisingly simple series representation

$$\prod_{m=1}^{\infty}(1 - x^m) = 1 + \sum_{n=1}^{\infty}(-1)^n\left[x^{\omega(n)} + x^{\omega(-n)}\right] = \sum_{n=-\infty}^{\infty}(-1)^n x^{\omega(n)}$$
$$= 1 - x - x^2 + x^5 + x^7 + \cdots , \quad (3.4.13)$$

where the integers $\omega(n) = \frac{1}{2}\left(3n^2 - n\right)$ are called the *pentagonal numbers* which can be illustrated geometrically as the numbers of equidistant points in a pentagon of increasing size (see Figure 3.3). These points form a pentagonal lattice. Thus, we state the *Euler Pentagonal Number Theorem*: If $|x| < 1$, then (3.4.13) holds. This is one of the celebrated discoveries of Euler in number theory. It is important to point out that generating functions usually play a fundamental role in probability and statistics and in the theory of elliptic and associated functions.

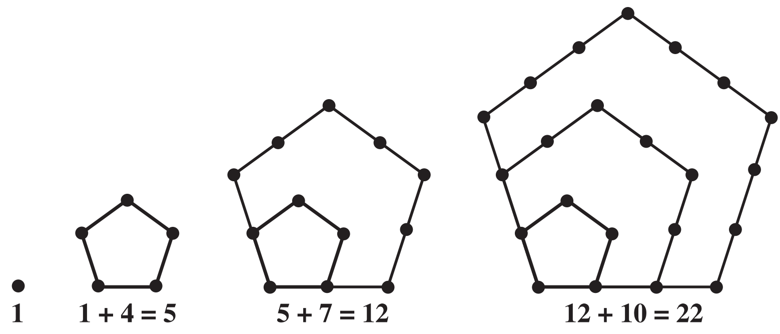


Fig. 3.3 The Pentagonal numbers, $\omega(n) = \frac{1}{2}n(3n - 1)$.

Furthermore, combining (3.4.2) and (3.4.13) leads to a recursion formula for the partition $p(N)$

$$p(N) = [p(N-1) + p(N-2)] - [p(N-5) + p(N-7)] + [p(N-12) + p(N-15)] - \cdots + (-1)^n p(N - \omega(-n)), \quad (3.4.14)$$

where $\omega(-n) = \frac{1}{2}n(3n+1)$. This relation (3.4.14) provides an efficient algorithm for computing the value of $p(N)$ for a given N. As $\frac{1}{2}(3n^2+n)$ assumes the values 0, 1, 2, 5, 7, 12, 15, $\cdots$ in succession, $n's$ are ticked off in sequence: 0, -1, 1, -2, 2, -3, 3 and so on. The series on the right hand side of (3.4.14) breaks off before the first n for which $\frac{1}{2}(3n^2+n) > N$. For example, given that $p(0) = 1$, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(5) = 7$, $p(6) = 11$ and $p(7) = 15$, we can use (3.4.14) to compute $p(8)$, $p(9)$, $p(10)$, $p(11)$ and $p(12)$ as follows:

$$\begin{aligned} N &= 8, n = 0, -1, 1, -2, 2, \\ p(8) &= p(8-1) + p(8-2) - p(8-5) - p(8-7) \\ &= p(7) + p(6) - p(3) - p(1) = 22 \\ p(9) &= p(8) + p(7) - p(4) - p(2) = 30. \end{aligned}$$

Similarly, we obtain $p(10) = 42$ and $p(11) = 56$. Finally, when $N = 12$, $n = 0, -1, 1, -2, 2, -3$, we find

$$p(12) = p(11) + p(10) - p(7) - p(5) + p(0) = 77.$$

The relation (3.4.14) was used to calculate

$$p(200) = 3972999029388. \quad (3.4.15)$$

Actual numerical computation reveals that the partition function $p(n)$ grows very rapidly with n. This leads to the question of exact or asymptotic representation of $p(n)$ for large n. During the early part of the 20th century, G. H. Hardy (1877-1947) and Srinivasa Ramanujan (1887-1920) made significant progress in the determination of an asymptotic formula for $p(n)$. Using elementary arguments, they first showed

$$\log p(n) \simeq \pi \left(\frac{2n}{3}\right)^{1/2} + O\left(\sqrt{n}\right) \quad \text{as} \quad n \to \infty. \quad (3.4.16)$$

Then, with the aid of a Tauberian Theorem, Hardy and Ramanujan (1918) proved that the number $p(n)$ of distinct ways of writing n as the sum of positive integers has the asymptotic representation:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left[\pi \left(\frac{2n}{3}\right)^{1/2}\right] \quad \text{as} \quad n \to \infty. \quad (3.4.17)$$

They also proved the following asymptotic formula for ${}_1p(n)$ which represents the number of distinct partition of n:

$$ {}_1p(n) \sim \frac{1}{4 \cdot 3^{\frac{1}{4}} n^{\frac{3}{4}}} \exp\left(\pi \sqrt{\frac{n}{3}}\right), \quad \text{as} \quad n \to \infty. \tag{3.4.18} $$

These are the most remarkable results in the theory of numbers. This asymptotic formula (3.4.17) can be used to calculate $p(200)$ which is an excellent agreement with the result (3.4.15). Equally remarkable was Hardy and Ramanujan's proofs of (3.4.17). One proof is based on the elementary recurrence relation

$$ p(n) = \frac{1}{2} \sum_{k=1}^{\infty} \sigma(k)\sigma(n-k), \qquad p(0) = 1, \tag{3.4.19} $$

where $\sigma(k)$ is the sum of the divisors of k. The asymptotic approximation of $\sigma(n)$ led to this result (3.4.17). Hardy and Ramanujan's second proof was based upon the Cauchy integral formula, which follows from (3.4.1) and Taylor's series expansion,

$$ p(n) = \frac{1}{2\pi i} \int_C \frac{F(z)}{z^{n+1}}\, dz, \tag{3.4.20} $$

where the function $F(z)$ is defined in (3.4.2) and it is analytic in the unit disk $|z| = 1$, and C is a closed contour enclosing the origin and lying entirely inside the unit disk. Finally, they proved that $F(x)$ in (3.4.2) is essentially a *modular form*. Making the change of variable $x = \exp(2\pi i\tau)$ the denominator $F(x)$ differs only by a simple factor from

$$ \eta(\tau) = \exp\left(\frac{i\pi\tau}{12}\right) \sum_{m=1}^{\infty} \{1 - \exp(2\pi i m\tau)\} = \frac{\exp\left(\frac{i\pi\tau}{12}\right)}{\sum_{n=0}^{\infty} p(n)\exp(2\pi i n\tau)}. \tag{3.4.21} $$

This is actually a modular form. In his famous series of 'Partitio Numerorum', Hardy and Littlewood devised a new remarkable technique, the so-called *Hardy–Littlewood circle method*, to obtain some new striking results. This method was very useful for the investigation of other additive questions in number theory. An explicit formula for the Fourier coefficients $p(n)$ of $[\eta(\tau)]^{-1}\exp(\pi i\tau/12)$ was found by using the circle method.

Using the modular character of $F(z)$, Hardy and Ramanujan applied the general theory of residues to $F(z)$ to obtain a series representation of $p(n)$. It has been shown that the rigorous proof of (3.4.19) depends on the Dedekind function and its behavior under the transformation of the modular group

$$ \tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1. \tag{3.4.22} $$

All these results seem to be very important in their own right and their proofs have successfully been generalized to deal with general modular forms of positive dimension, stimulating a vast amount of research in the theory of modular functions during the twentieth century.

Subsequently, Hardy and Ramanujan obtained a truly remarkable result in the form

$$p(n) \sim \sum_{q=1}^{n} L_q(n)\Phi_q(n) \tag{3.4.23}$$

where

$$\Phi_q(n) = \frac{\sqrt{q}}{2\pi\sqrt{2}}\,\frac{d}{dn}\lambda_n^{-1}\exp\left(\left(\frac{2}{3}\right)^{1/2}\lambda_n\right), \qquad \lambda_n = \left(n - \frac{1}{24}\right)^{1/2}, \tag{3.4.24ab}$$

$$L_q(n) = \sum_q \omega_{p,q}\exp(-2\pi ip/q), \tag{3.4.25}$$

p runs through the integers less than and prime to q, and $\omega_{p,q}$ is a certain $24q$th roots of unity.

In particular,

$$p(n) \sim \frac{1}{4\sqrt{3}\,\lambda_n^2}\exp(K\lambda_n) + O\left(\frac{\exp(K\lambda_n)}{\lambda_n^3}\right), \quad K = \pi\left(\frac{2}{3}\right)^{1/2}, \ n \geq 1. \tag{3.4.26}$$

Moreover, when λ_n^2 is replaced by n, then (3.4.26) becomes identical with (3.4.17). Finally, in 1937, Hans Rademacher (1892-1969) further improved and fully completed the asymptotic analysis of $p(n)$ by proving an exact formula

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{q=1}^{\infty}\sqrt{q}\,A_q(n)\,\frac{d}{dn}\left[\frac{1}{\lambda_n}\sinh\left(\frac{K\lambda_n}{q}\right)\right], \qquad n \geq 1, \tag{3.4.27}$$

where

$$A_q(n) = \sum_{p \bmod q}\omega_{p,q}\exp(-2n\pi ip/q), \quad \omega_{p,q} = \exp[\pi i\, s(p,q)], \quad (p,q) = 1, \tag{3.4.28}$$

and $s(p,q)$ is the Dedekind sum.

Thus, the theory of Hardy–Ramanujan's partitions as well as the work of Redemacher is truly remarkable and has stimulated tremendous interests in subsequent developments in the theory of modular functions. The Hardy–Ramanujan collaboration on the asymptotic formula for $p(n)$ produced one of the monumental results in the history of mathematics.

Ramanujan made some significant contributions to the theory of partitions. He was not only the first but the only mathematician who successfully proved several remarkable congruence properties of $p(n)$. Some of his congruences include

$$p(5m+4) \equiv 0 \pmod{5}, \tag{3.4.29}$$

$$p(7m+5) \equiv 0 \pmod{7}, \tag{3.4.30}$$

$$p(11m+6) \equiv 0 \pmod{11}. \tag{3.4.31}$$

All these results are included in his famous conjecture: If $p = 5$, 7 or 11 and $24n - 1 \equiv 0 \pmod{p^\alpha}$, $\alpha \geq 1$, then

$$p(n) \equiv 0 \pmod{p^\alpha}. \tag{3.4.32}$$

This was a very astonishing conjecture and has led to a good deal of theoretical research and numerical computation on congruence of $p(n)$ using H. Gupta's table (1980) of values of $p(n)$ for $n \leq 300$. However, S. Chowla (1907-1995) found that this conjecture is not true for $n = 243$. For this n, $24n - 1 = 5831 \equiv 0 \pmod{7^3}$ but

$$\begin{aligned} p(243) = 133978259344888 &\equiv 0 \pmod{7^2}, \\ &\not\equiv 0 \pmod{7^3}. \end{aligned} \tag{3.4.33}$$

Subsequently, in 1936, D. H. Lehmer (1905-1991) became deeply involved in the proof of the conjecture and also in the computation of $p(n)$ for large n. In 1938, G. N. Watson (1907-1995) proved Ramanujan's conjecture for powers of 7. Finally, in 1967, A. O. L. Atkins (1925-2008) settled the problem by proving the conjecture for powers of 11. Ramanujan's conjecture can now be stated as an important theorem: If $24n - 1 \equiv 0 \pmod{d = 5^a 7^b 11^c}$, then

$$p(n) \equiv 0 \pmod{d}. \tag{3.4.34}$$

One of the remarkable applications of the theory of partitions deals with the problems of statistical mechanics. The central problem of statistical mechanics is the determination of number of ways a given amount of energy can be shared out among the different possible states of an assembly. This problem is essentially the same type as that of finding the number of partitions of a number into integers under certain restrictions. The methods of partitions have been applied to study the Bose–Einstein condensation of a perfect gas. Several authors including Auluck and Kothari (1946), Temperley (1949) and Dutta (1956) have discussed the significant role of partition functions in statistical mechanics.

We consider an assembly of N non-interacting identical linear simple harmonic oscillators. The energy levels associated with an oscillator are $\varepsilon_m = (m + \frac{1}{2})\hbar\omega$ where m is a non-negative integer, $h = 2\pi\hbar$ is the *Planck constant*, and ω is the angular frequency. If E represents the energy of the assembly, a number n, in units of $\hbar\omega$, is defined by

$$\hbar\omega n = E - \frac{1}{2}N\hbar\omega \tag{3.4.35}$$

where the second term on the right hand side represents the residual energy of the oscillators. We denote $\psi(E)$ for the number of distinct wave functions assigned to the assembly for the energy state E. It is well known that, for a Bose–Einstein assembly, the number of assigned wave functions is the number of ways of distributing n energy quanta among N identical oscillators without any restriction as to the number of quanta assigned to the oscillator. For a Fermi–Dirac assembly, the energy quanta assigned to all oscillators are all different. For the case of a classical Maxwell–Boltzmann assembly, oscillators are considered as distinguishable from each other, and the number of wave functions is simply the number of ways of distributing n energy quanta among N distinguishable oscillators which is equal to the number of ways of assigning N elements to n positions, repetitions of any element are permissible.

It turns out that

$$\psi(E) = p_N(n) \quad \text{for the Bose–Einstein assembly,} \tag{3.4.36}$$

$$\psi(E) = Q_N(n) + Q_{N-1}(n) = Q_n(n+N) \text{ for the Fermi–Dirac assembly,} \tag{3.4.37}$$

$$\psi(E) = \frac{{}^N H_n}{N!} = \frac{(N+n-1)!}{N!\,(N-1)!\,n!} \quad \text{for the Maxwell–Boltzmann assembly,} \tag{3.4.38}$$

where $\frac{1}{N!}$ is inserted to make the entropy expression meaningful.

It is interesting to point out that when $N = O(\sqrt{n})$, $p_N(n)$ and $Q_N(n+N)$ tend to ${}^N H_n/N!$. This means that, for $N \ll \sqrt{n}$, both the Bose–Einstein statistics and the Fermi–Dirac statistics tend to the classical Maxwell–Boltzmann statistics.

It has been confirmed that the results of the Bose–Einstein condensation phenomena are in excellent agreement with those obtained by using the Hardy–Ramanujan asymptotic formula. This shows the great importance of the Hardy–Ramanujan asymptotic result in statistical mechanics. A method similar to that employed for the derivation of the results for the Bose–Einstein assembly can be used successfully to derive asymptotic formula for the Fermi–Dirac assembly. Thus, any thermodynamic assembly of non-interacting particles can be described by the Hardy–Ramanujan partition formula. Many interesting results for the Bose–Einstein condensation theory have also been obtained by using the properties of partition functions.

In essentially statistical approach to thermodynamic problems, Dutta (1955) obtained some general results from which different statistics viz., those of Bose–Einstein, Fermi–Dirac and Gentile, Maxwell–Boltzmann can be derived by using different partitions of numbers. It is noted that mathematical problems of statistics of Bose–Einstein, Fermi–Dirac and Gentile are those of partitions of numbers (energy) into partitions in which repetition of parts are restricted differently. In partitions corresponding to Bose statistics any part can be repeated any number of times ($d \to \infty$), that to Gentile statistics any part can be repeated up to d times where d is a fixed positive integer, and that to Fermi statistics no part is allowed to repeat, that is, $d = 1$. All these led to an investigation of a new and different type of partitions of numbers in which repetition of any part is restricted suitably. Motivated by the need of such partition functions and its physical applications to statistical physics, Dutta (1956, 1957) studied a new partition of number n into any number of parts, in which no part is repeated more than d times. Dutta's partition function is devoted by ${}_dp(n)$. Dutta himself and in collaboration with Debnath proved algebraic and congruence properties of ${}_dp(n)$. They obtained a simple algebraic formula to calculate successively the numerical values of ${}_dp(n)$ from the values of $p(n)$ and so ultimately from Euler's table. Using a Tauberian theorem, they also proved an asymptotic formula correct up to the exponential order for ${}_dp(n)$ as

$$ {}_dp(n) \sim \exp\left[\pi\left\{\frac{2}{3}n\left(\frac{d}{d+1}\right)\right\}^{1/2}\right] \quad \text{as} \quad n \to \infty. \tag{3.4.39}$$

For partitions into unequal parts ($d = 1$) and for unrestricted partitions ($d \to \infty$), the above result reduces to the Hardy–Ramanujan formulas up to the exponential order. This formula (3.4.39) for the unrestricted partitions is found to be very useful to determine the dominating term in the expression for entropy of the corresponding thermodynamic system.

Subsequently, Dutta and Debnath (1959) also introduced a new partition function ${}_dp(n/m)$ representing the number of partitions of an integer n into m parts with at most d repetitions of each part. They proved the generating function and congruence properties with examples. Several special cases of this partition functions are also discussed with examples. Finally, it may be pertinent to mention that several authors (see Gupta (1980)) have also studied new types of partition functions and their properties for possible applications.

3.5 Euler's Contributions to Continued Fractions

Historically, continued fractions first occurred in ancient arithmetic in connection with the approximation of irrational numbers by rational numbers. They also originated from the Euclid algorithm for finding the greatest common divisor of two integers a and $b(a > b)$ so that $a = a_0 b + r_0$, where a_0 is the quotient and r_0 is the remainder. This can be expressed in the form

$$\frac{a}{b} = a_0 + \frac{1}{\frac{b}{r_0}} = a_0 + \frac{1}{a_1 + \frac{r_1}{r_0}} = a_0 + \frac{1}{a_1 + \left(\frac{r_0}{r_1}\right)^{-1}}. \tag{3.5.1}$$

The continuation of the Euclid algorithm leads to the continued fraction

$$\frac{a}{b} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \tag{3.5.2}$$

which is also written in the form

$$\frac{a}{b} = a_0 + \frac{1}{a_1}\,\frac{1}{a_2}\,\frac{1}{a_3}\,\cdots = [a_0; a_1, \cdots a_n, \cdots]. \tag{3.5.3}$$

An expression (finite or infinite) on the right hand side of (3.5.2) or (3.5.3) is called a *simple continued fraction*, where a_0, a_1, a_2, $\cdots$, a_n, $\cdots$ are real or complex numbers with a_0 may be zero or non-zero, but all $a_n \neq 0$, $n \geq 1$. The first few *partial fractions* or *convergents* of (3.5.3) are $c_0 = a_0$, $c_1 = [a_0; a_1] = a_0 + \frac{1}{a_1}$, $c_2 = [a_0; a_1, a_2] = a_0 + \frac{1}{\left(a_1 + \frac{1}{a_2}\right)} = a_0 + \frac{1}{[a_1, a_2]}$, and in general,

$$c_n = [a_0; a_1, a_2, \cdots, a_n] = a_0 + \frac{1}{[a_1, a_2, \cdots, a_n]}.$$

An iteration process can be used to define

$$p_k = a_k p_{k-1} + p_{k-2} \qquad \text{and} \qquad q_k = a_k q_{k-1} + q_{k-2}, \quad k = 0, 1, 2, \cdots, n, \tag{3.5.4}$$

where $p_{-2} = 0$, $p_{-1} = 1$, $q_{-2} = 1$, $q_{-1} = 0$ so that

$$\frac{p_n}{q_n} = [a_0; a_1, \cdots, a_n], \qquad n = 0, 1, 2, \cdots. \tag{3.5.5}$$

An infinite continued fraction $[a_0; a_1, a_2, \cdots, a_n, \cdots]$ is said to be *convergent* if

$$\lim_{n\to\infty} \frac{p_n}{q_n} = a \tag{3.5.6}$$

exists and a is called the *value* of the infinite continued fraction.

It can be shown that every real number a can be represented uniquely by a continued fraction. In fact, every finite continued fraction represents a rational number and every infinite continued fraction represents an irrational number. For example, the irrational number $\sqrt{2}$ can be represented by an infinite continued fraction. We write

$$\sqrt{2} = 1 + \frac{1}{x}, \quad \text{or} \quad x = 2 + \sqrt{2} - 1 = 2 + \frac{1}{x} \tag{3.5.7}$$

so that

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{x}} = 1 + \frac{1}{2 + \frac{1}{2+\frac{1}{x}}}. \tag{3.5.8}$$

This process can be continued to generate the simple continued fraction for $\sqrt{2}$ as

$$\sqrt{2} = [1; 2, 2, \cdots], \tag{3.5.9}$$

with the first few partial fractions

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \cdots.$$

In his famous 1655 book on *Arithematica Infinitorum*, the British mathematician, John Wallis gave an infinite product representation of $\frac{4}{\pi}$ as

$$\frac{4}{\pi} = \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \frac{7}{8} \cdots. \tag{3.5.10}$$

He also stated in his book that Lord William Brouncker (1620-1684), the first President of the Royal Society of London has expressed the product (3.5.10) into continued fraction without proof in the form

$$\frac{4}{\pi} = 1 + \frac{1}{2+} \frac{9}{2+} \frac{25}{2+} \frac{49}{2+\cdots}. \tag{3.5.11}$$

Any expression of the form

$$a_0 + \frac{b_1}{a_{1+}} \frac{b_2}{a_{2+}} \frac{b_3}{a_{3+}} \cdots \frac{b_n}{a_{n+}} \cdots \tag{3.5.12}$$

is called the *general continued fraction*. The first few convergents are

$$c_0 = a_0, \quad c_1 = a_0 + \frac{b_1}{a_1}, \quad c_2 = a_0 + \frac{b_1}{a_{1+}} \frac{b_2}{a_2} \cdots,$$

$$c_n = a_0 + \frac{b_1}{a_{1+}} \frac{b_2}{a_{2+}} \cdots \frac{a_n}{b_n}.$$

Based on earlier work of his predecessors, Euler began his research on continued fractions and published many new ideas and results in his first paper entitled, "*De Fractionibus Continuis*" in 1737. He also proved that any rational number can be represented by a finite continued fraction and found an infinite continued fraction representation for e in the following form

$$e = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+\cdots},$$

$$= [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \cdots], \tag{3.5.13}$$

with the first few partial fractions

$$\frac{2}{1},\ \frac{3}{1},\ \frac{8}{3},\ \frac{11}{4},\ \frac{19}{7},\ \frac{87}{32},\ \cdots.$$

Or, equivalently, decimal representations of these fractions are

$$2.00,\ 3.00,\ 2.666\cdots,\ 2.75,\ 2.71428\cdots,\ 2.71875,\ \cdots,$$

so that the approximations get better and better. A continued fraction representation of $(e+1)/(e-1)$ is

$$\frac{e+1}{e-1} = 2 + \frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+\cdots}. \tag{3.5.14}$$

Both (3.5.13) and (3.5.14) represent infinite continued fractions. Euler proved that e and e^2 are irrational as they can be represented by infinite continued fractions. He also gave many continued-fractions representations of both rational and irrational numbers. His continued fractions for π were

$$\pi = 3 + \frac{1}{6+}\,\frac{9}{6+}\,\frac{25}{6+}\,\frac{49}{6+\cdots}, \tag{3.5.15}$$

$$\frac{\pi}{2} = 1 + \frac{1}{1+}\,\frac{1.2}{1+}\,\frac{2.3}{1+}\,\frac{3.4}{1+\cdots}, \tag{3.5.16}$$

$$\frac{\pi}{2} = 1 + \frac{2}{3+}\,\frac{1.3}{4+}\,\frac{3.5}{4+}\,\frac{5.7}{4+\cdots}. \tag{3.5.17}$$

In 1768, Lambert found a simple but irregularly behaved continued fraction representation for π as

$$\pi = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{292+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{14+\cdots},$$

$$= [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, \cdots], \tag{3.5.18}$$

with the first few partial fractions

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \cdots .$$

Euler proved another remarkable formula

$$\frac{(\sqrt{e})^n + 1}{(\sqrt{e})^n - 1} = [n; 3n, 5n, \cdots], \qquad n = 1, 2, 3, \cdots . \tag{3.5.19}$$

He also proved a theorem which states that a root of a quadratic equation is real if and only if it has a periodic continued fraction representation.

For example $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{7}$ have infinite periodic continued fraction representations in the form $\sqrt{2} = [1; \overline{2}]$, $\sqrt{3} = [1; \overline{1, 2}]$, $\sqrt{5} = [2; \overline{4}]$, $\sqrt{7} = [2; \overline{1, 1, 1, 4}]$ and $\sqrt{10} = [3; \overline{6}]$, where the period is indicated by overbar line.

On the other hand, the *golden ratio* $x = \frac{1}{2}(\sqrt{5} - 1)$, where $x^2 + x - 1 = 0$ has the simplest periodic continued fraction representation

$$x = \frac{1}{2}(\sqrt{5} - 1) = [0; 1, 1, 1, 1, \cdots] = [0; \overline{1}] \tag{3.5.20}$$

with the first few partial fractions

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \cdots .$$

Similarly, the continued fraction of the golden ratio x is

$$x = \frac{1}{2}(\sqrt{5} + 1) = [1; 1, 1, 1, \cdots] = [\overline{1}], \tag{3.5.21}$$

with the first few partial fractions

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \cdots .$$

It follows from (3.5.5) that the approximate fractions (or convergents) of a continued fraction for $n = 0, 1, 2, \cdots$, are

$$\frac{p_n}{q_n} = [a_0],\ [a_0; a_1],\ [a_0; a_1, a_2], \cdots [a_0; a_1, \cdots a_n] \cdots . \tag{3.5.22}$$

It can be shown that the best rational approximation of a real number x satisfies the error estimate

$$\left| x - \frac{p_n}{q_n} \right| \le \frac{1}{q_n\, q_{n+1}}, \tag{3.5.23}$$

where the error depend on q_n.

For example, $\sqrt{2} = [1, \overline{2}]$ which has the first few partial fractions

$$\frac{p_n}{q_n} = 1,\ \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \cdots . \tag{3.5.24}$$

It follows that $\frac{17}{12}$ is the best rational approximation of $\sqrt{2}$ with a denominator ≤ 12 with the error estimate (3.5.23) given by

$$\left|\sqrt{2} - \frac{17}{21}\right| = \left|\sqrt{2} - \frac{p_3}{q_3}\right| \leq \frac{1}{q_3\, q_4} = \frac{1}{12 \times 29} < 3\left(10^{-3}\right).$$

For $e = [2; 1, 2, 1, 1, 4, 1, 1, \cdots]$, the first few partial fractions are

$$\frac{p_n}{q_n} = \frac{2}{1}, \frac{3}{1}, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}, \frac{106}{39}, \cdots. \tag{3.5.25}$$

Then $\frac{87}{32}$ is the best rational approximation for e with a denominator ≤ 32 and the error estimate (3.5.23) is

$$\left|e - \frac{87}{32}\right| = \left|e - \frac{p_5}{q_5}\right| \leq \frac{1}{q_5\, q_6} = \frac{1}{32 \times 39} < 10^{-3}.$$

Similarly, the golden ratio $x = \frac{1}{2}\left(\sqrt{5} - 1\right) = [0, \overline{1}]$ has a few approximate fractions

$$\frac{p_n}{q_n} = \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \cdots. \tag{3.5.26}$$

The best rational approximation of x is $\frac{8}{13}$ with a denominator ≤ 13 with the error estimate

$$\left|x - \frac{8}{13}\right| = \left|x - \frac{p_6}{q_6}\right| \leq \frac{1}{q_6\, q_7} = \frac{1}{13 \times 21} < 4\left(10^{-3}\right).$$

For $\pi = [3; 7, 15, 1, 292, 1, 1, 1, \cdots]$ which has the first few partial fractions

$$\frac{p_n}{q_n} = \frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \cdots. \tag{3.5.27}$$

From the time of Archimedes, it is universally believed that $\frac{22}{7}$ is the best possible approximation of π as it occurred in the calculation of the area and the perimeter of a circle. Indeed, it follows from the error estimate (3.5.23) that $\frac{22}{7}$ is the best possible rational approximation of π provided the denominator of the fraction ≤ 7. Surprisingly, a Chinese mathematician Zu Chong–Zhi's (A.D. 430–501) work contained $\frac{355}{113}$ as the best approximation of π if the denominator ≤ 113 and the error estimate is

$$\left|\pi - \frac{355}{113}\right| \leq \frac{1}{q_3\, q_4} = \frac{1}{113 \times 33102} < 10^{-6}.$$

This means that $\frac{355}{113}$ is the best possible rational approximation of π, but $\frac{22}{7}$ is *not the best possible accurate* value of π.

The Euler constant γ has a continued fraction expansion

$$\gamma = [0; 1, 1, 2, 1, 2, 1, 4, 3, \cdots] \tag{3.5.28}$$

with the first few partial fractions

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{3}{5}, \frac{4}{7}, \frac{11}{19}, \frac{15}{26}, \cdots .$$

The determination of the optimal rational approximation of a real number has an important and useful role in the theory of approximations. Several optimal approximation theorems have been developed by many great mathematicians. According to a 1958 Fields Prize Winner in Mathematics, Klaus Roth (1925-) algebraic irrotational number can only be poorly approximated by rational numbers, but transcendental numbers can be accurately approximated by rational numbers. In 1955, he proved a celebrated fundamental approximation theorem on algebraic irrational numbers by rationals.

Euler was the first mathematician who showed how to transform an infinite series to a continued fraction representation of the series and conversely. In his 1754 paper on divergent series and his correspondence with Nicholas Bernoulli, Euler first proved that the series

$$y = x - (1!)x^2 + (2!)x^3 - (3!)x^4 + \cdots \tag{3.5.29}$$

formally satisfies the ordinary differential equation

$$x^2 y' + y = x. \tag{3.5.30}$$

He then obtained the integral solution of (5.3.30) in the form

$$y = \int_0^x \frac{xe^{-t}}{1+xt} dt. \tag{3.5.31}$$

He went further to develop rules for transformation of series (3.5.29) into a continued fraction representation in the form

$$\frac{x}{1+} \, \frac{x}{1+} \, \frac{x}{1+} \, \frac{2x}{1+} \, \frac{2x}{1+} \, \frac{3x}{1+} \, \frac{3x}{1+\cdots}. \tag{3.5.32}$$

He then substituted $x = 1$ in the continued fraction (3.5.32) to calculate a value of the divergent series

$$1 - 2! + 3! - 4! + 5! - \cdots . \tag{3.5.33}$$

3.6 Euler's Contributions to Classical Algebra

Around 250 A.D., an ancient Greek mathematician, Diophantus of Alexandria became very famous for his great and highly original works, *Arithematica*. It dealt with an analytical treatment of algebraic number theory and solutions of equations which are known as the *Diophantine equations*. In addition, *Arithematica* contained many new theorems concerning the representation of numbers as the sum of two, three or four squares. Fermat, Euler and Lagrange made considerable investigations of these representation problems, and Diophantine equations which dealt with solutions in positive integers or rational numbers. In 1770, Euler published two-volume of classical algebra entitled *Vollstándige Anleitung zur Algebra* in which many fundamental ideas and results of classical algebra have been included. In addition, he investigated Diophantine equations and the algebraic theory of numbers. In his work on algebra, Euler not only developed new concepts and methods, but also introduced the ideas of birational equivalence of curves over the field Q of rational numbers, and quite new results in the arithmetic of elliptic curves.

Euler's major interest in algebra originated from the *Fermat Diophantine equations* (3.3.9). It is well known that Fermat claimed that the equation (3.3.9) *has no solutions in integers* x, y and z for $n > 2$. Many great mathematicians since Fermat's time have made serious attempt to prove the Fermat Last Theorem, but they were unsuccessful. It is a great delight to quote David Hilbert's statement presented at the 1900 second International Congress of Mathematicians in Paris as follows:

"It is well known that Fermat claimed that the Diophantine equation

$$x^n + y^n = z^n$$

- with trivial exceptions - has no solutions in integers x, y, z. The problem of showing this non-solvability result gives an excellent example of how a special and seemingly meaningless problem can give incredible impetus to scientific research. In fact, roused by the challenge of this Fermat conjecture, Kummer was led to his introduction of ideal numbers and to the discovery of the theorem of the unique decomposition of numbers of a cyclotomic field into ideal prime factors — a theorem which, in the form of the generalization of the result due to Dedekind and Kronecker to general algebraic systems, is at the heart of modern number theory and has importance far beyond the boundaries of number theory in the areas of algebra and function theory."

It may be interesting to point out that, for $n = 2$, equation (3.3.9) reduces to the famous *Pythagorean equation* $x^2 + y^2 = z^2$ which has infinite number of integral solutions including $(3, 4, 5)$, $(5, 12, 13)$, $(8, 15, 17)$, $(12, 35, 37)$. More generally, (x, y, z) is a *primitive Pythagorean triple* if and only if

$$(x, y, z) = (r + t, \quad s + t, \quad r + s + t), \tag{3.6.1}$$

where r, s, t are some integers satisfying $gcd\,(r, s) = 1$ and $t^2 = 2rs$. It can be shown that if x, y, z are all positive, then r, s, t are also positive and vice versa. On the other hand, if (x, y, z) is a given Pythagorean triple, the related integers r, s, t are given by

$$r = z - y, \quad s = z - x, \quad t = x + y - z. \tag{3.6.2}$$

Conversely, there are infinitely many pairs of consecutive positive integers such that one is an odd square and the other is twice a square. It was proved by Fermat that there are infinitely many Pythagorean triples (x, y, z) such that $x-y = \pm 1$. So, the related r and s satisfying $r-s = x-y = \pm 1$ can be generated with the above requirements. It turns out that the Pythagorean triples with positive entries must be of the form (3.6.1) with $r = 2u^2$ and $s = v^2$ (or vice versa) for some u, $v \in \mathbb{Z}$ with odd v. Furthermore, $t^2 = 2rs = (2uv)^2$ and so, $t = 2uv$. Putting $m = u + v$ and $n = u$ gives the standard representation of primitive Pythagorean triples of the form

$$(x, y, z) = (2mn,\ m^2 - n^2,\ m^2 + n^2) \tag{3.6.3}$$

provided $(m, n) = 1$ and $0 < n < m$. If x, y, and z are relatively prime integers such that $x^2 + y^2 = z^2$ with y and z odd and x even, there exist integers m and n such that $x = 2mn$, $y = m^2 - n^2$ and $z = m^2 + n^2$. This can be proved by writing $x^2 = z^2 - y^2 = (z + y)(z - y) = m^2n^2$ for some m and n so that $z + y = m^2$ and $z - y = n^2$. This gives $2z = m^2 + n^2$ and $2y = m^2 - n^2$ and hence, $(2x)^2 = (2z)^2 - (2y)^2 = (2mn)^2$ and then $2x = 2mn$. There are similar representations of $x^2 + y^2 = z^2$, where $x = (2n^2 + 2n)$, $y = (2n + 1)$ and $z = (2n^2 + 2n + 1)$, and $x = 2m$, $y = (m^2 - 1)$ and $z = (m^2 + 1)$.

Another related question is the area of the right-angled triangle with sides x, y and z given by

$$\triangle = \frac{1}{2}\, xy = mn(m + n)(m - n). \tag{3.6.4}$$

Since one of m and n must be even as $m + n$ is odd, and one of m, n, $(m^2 - n^2)$ must be divisible by 3, the area $\triangle$ must be divisible by 6. So, the question is how few prime factors can $(\triangle/6)$ have? The area $\triangle$ given by (3.6.4) is the product of four factors which are linear polynomials in m and n. Thus, there can only be a finite number of pairs m and n for which $(\triangle/6)$ has fewer than three prime factors. For the triangle, $(3, 4, 5)$, $(\triangle/6) = 1$, and for the triangle, $(5, 12, 13)$, $(\triangle/6) = 5$ which has only one prime factor. For the only triangles, $(8, 15, 17)$, $(7, 24, 25)$, $(12, 35, 37)$, $(20, 21, 29)$, $(11, 60, 61)$, and $(13, 84, 85)$, $(\triangle/6)$ has exactly two prime factors. According to Granville (2008), there are infinitely many Pythagorean triples for which $(\triangle/6)$ has exactly three prime factors. This has not yet been proved.

If P is a perfect number, then there exist positive integers x, y and z such that $x < y < z$ and $P = x + y + z$ and $(x + y,\ x + z,\ y + z)$ is a Pythagorean triple. In other words,

$$(x + y)^2 + (x + z)^2 = (y + z)^2. \tag{3.6.5}$$

According to result (3.6.3), $(x+y,\ x+z,\ y+z)$ would be a Pythagorean triple provided that there exist relatively prime positive integers m and $n\,(0 < n < m)$ such that

$$(x + y,\ x + z,\ y + z) = (2mn,\ m^2 - n^2,\ m^2 + n^2). \tag{3.6.6}$$

Consequently,

$$x = n(m - n), \quad y = n(m + n), \quad z = m(m - n), \tag{3.6.7}$$

and $P = x + y + z = m(m + n)$.

Since P is an even perfect number, $P = M_p \cdot 2^{p-1}$, where $M_p = 2^p - 1$ is a Mersenne prime and p is a prime. So, $P = (2^p - 1)2^{p-1} = m(m + n)$. Since m and n are relatively primes, $m = 2^{p-1}$ and $m + n = 2^p - 1$ or $n = 2^{p-1} - 1$. Thus,

$$x = (2^{p-1} - 1), \quad y = (2^p - 1)(2^{p-1} - 1), \quad z = 2^{p-1}. \tag{3.6.8}$$

This confirms that $(x + y,\ x + z,\ y + z)$ is a Pythagorean triple.

The next natural question is whether there are Pythagorean triples over the ring $\mathbb{Z}[i]$ of Gaussian integers. If (x, y, z) is a Pythagorean triple, then (x, iz, iy) and (z, ix, y) are also Pythagorean triples. More generally, there are many Pythagorean triples over $\mathbb{Z}[i]$ including $(x, y, z) = (1+2i, 2+i, 2+2i)$, $(2 + 2i, 2 - i, 2 + i)$, $(7 + 4i, 4 + i, 8 + 4i)$ and $(6 + 13i, 3 + 18i, 6 + 22i)$.

It is interesting to note that Fermat used the method of infinite descent to prove that there cannot be a Pythagorean triangle $\left(a^2 + b^2 = c^2\right)$ whose area $\triangle = \frac{1}{2}ab$ is the square of an integer. However, if there is a solution of (3.3.9) for $n = 4$, then $a^2 + b^2 = c^2$ and $\triangle = \frac{1}{2}ab = d^2$ hold. We assume that there is a solution of $x^4 + y^4 = z^4$. In terms of this solution, we set $a = y^4$, $b = 2x^2z^2$ and $c = z^4 + x^4$ and $d = xzy^2$ so that $a^2 + b^2 = c^2$ holds. Thus,

$$\triangle = \frac{1}{2}ab = x^2z^2y^4 = \left(xzy^2\right)^2 = d^2. \tag{3.6.9}$$

Hence the assumption that the equation $x^4 + y^4 = z^4$ has a solution must be false. So far, no example has yet been found.

Fermat rediscovered another diophantine equation $x^2 - ny^2 = 1$ or $x^2 - ny^2 = m$ where n and m are integers with n not a perfect square and proved that this equation has an infinite number of integer solutions (x, y). Based on Fermat's work on this equation, Euler erroneously named the equation as *Pell's equation*

$$x^2 - ny^2 = 1, \tag{3.6.10}$$

where n is not a perfect square, and worked extensively on this equation. He found the relation between the fundamental solution to the continued fraction of $\sqrt{n}$ as well as the period of the continued fraction for quadratic irrationalities. Euler also obtained the fundamental solutions for a wide range of numerical values of n. Indeed, the seventh century Indian Hindu mathematician, Brahmagupta showed that (x, y) is a solution of (3.6.10), then $\left(x^2 + ny^2,\ 2xy\right)$ is also a solution because $\left(x^2 + ny^2\right)^2 - n(2xy)^2 = \left(x^2 - ny^2\right)^2 = 1$.

Euler and Lagrange proved many special Diophantine equations and also showed that certain primes can be expressed in particular ways. In 1754, Euler proved the Fermat result that every prime of the form $4n + 1$ can be represented uniquely as a sum of two squares. He also proved that $x^4 + y^4 = z^2$ has no solution in positive integers. On the other hand, he showed that a prime of the form $3n + 1$ can be represented uniquely in the form $x^2 + 3y^2$. Clearly, some integers n can be expressed as the sum of two or three or four squares or even as just one square. However, in 1770, both Euler and Lagrange proved that any positive integer can be expressed as the sum of four squares, that is, $n = x^2 + y^2 + z^2 + w^2$. It is not necessary to use four squares in order to represent a positive integer n, but merely that four squares are always sufficient. For example,

$$\begin{aligned}
&15 = 1^2 + 2^2 + 3^2 + 1^2, \qquad 22 = 1^2 + 1^2 + 2^2 + 4^2,\\
&17 = 1^2 + 4^2 = 2^2 + 2^2 + 3^2,\\
&51 = 1^2 + 3^2 + 4^2 + 5^2 = 1^2 + 1^2 + 7^2 = 1^2 + 5^2 + 5^2,\\
&81 = 2^2 + 2^2 + 3^2 + 8^2 = 1^2 + 4^2 + 8^2 = 9^2.
\end{aligned}$$

The four square representation problem is a particular case of a more general representation problem known as the *Waring Problem (Conjecture)* formulated by Edward Waring, Lucasian Professor of Mathematics at the University of Cambridge, England. In his 1770 book entitled *Meditationes Algebraicae*, Waring stated that every positive integer n can be represented as the sum of at most k mth powers in the form

$$n = x_1^m + x_2^m + \cdots + x_k^m, \tag{3.6.11}$$

where $x_1, x_2, \cdots, x_k$ are nonnegative integers and $k = f(m)$. In other words, given a positive integer n and $m \geq 2$, there is a smallest positive integer $k = f(m)$ such that the Diophantine equation (3.6.11) has integral solutions $x_1, x_2, \cdots, x_k$. Evidently, it follows from the four square problem that $f(2) = 4$ (four squares). Waring made a conjecture about $f(3)$ and $f(4)$. Based on limited numerical evidence, he found $f(3) = 9$ (nine cubes). Waring did not provide any proof of his conjecture. In 1909, David Hilbert proved the Waring problem and showed also that $f(m)$ is finite for all m, but his proof provided no method how to compute $f(m)$. However, recent result gives the lower bound for $f(m)$ in the form

$$f(m) \geq (2^m - 2) + \left[\left(\frac{3}{2}\right)^m\right] \quad \text{for all } m \geq 2, \tag{3.6.12}$$

where $[m]$ is the standard symbol for the greatest integer $\leq m$. In 1992, it was proved that $f(4) = 19$ (19 fourth powers). In 1939, it was also proved that 23 and 239 are the only integers that require nine cubes. In 1943, Yu V. Linnik (1915-1972) proved that only finitely many integers require eight cubes. It seems that the Waring problem has not been completely solved, although some but slow progress has been made on this problem.

Euler made an extensive study of different types of Diophantine equations such as $y^2 = ax^3 + b^2$, $y^2 = ax^2 + bx + c$, where a, b, c, are integers with $a > 0$ and not a perfect square. More generally, he investigated equations of the form

$$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0, \tag{3.6.13}$$

where the discriminant $D^* = B^2 - AC > 0$ and not a square.

From the seventeenth century, the theory of equations dealing with the solution of polynomial equations was a major subject in classical algebra. The main interest was in developing better methods of solving equations of any degree, finding better methods of approximation of roots of equations and in proving the existence theorem that every nth degree polynomial equation has n roots. Many great mathematicians made contributions to this subject. Most notably among them are Euler and d'Alembert who spent considerable time and energy in discovering new ideas and results. In particular, Euler began his research with the idea that every polynomial with real coefficients can be decomposed into product of linear and quadratic factors with real coefficients. The solution of quadratic equation $ax^2 + bx + c = 0$ by the method of completing the square had been known since ancient times, and the only progress in this equation until

1500 was made by the Indian mathematicians. They developed the celebrated method for finding the roots α and β of the quadratic equation $ax^2 + bx + c = 0$ as follows:

$$0 = ax^2 + bx + c = a(x-\alpha)(x-\beta) = a\left[x^2 - (\alpha+\beta)x + \alpha\beta\right], \quad (3.6.14)$$

where α and β are related by the formulas

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}. \quad (3.6.15)$$

Using the so called *polarization identity*

$$4\alpha\beta = (\alpha+\beta)^2 - (\alpha-\beta)^2, \quad (3.6.16)$$

formulas (3.6.15) are replaced by the linear relations

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha - \beta = \pm\frac{1}{a}\sqrt{b^2 - 4ac}, \quad (3.6.17)$$

so that the roots are given by the celebrated formulas

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (3.6.18)$$

Thus, this classical approach to solving the quadratic equation required the four basic algebraic operations of addition, subtraction, multiplication, division and the square root extraction which is called *solution by radicals*. It is noted here that although the roots of the quadratic equation appear symmetrically in the equation, but the symmetry is broken due to the square root taken in the polarization identity. Thus, the solution of a quadratic equation leads to a linear system of equations (3.6.17) for its roots α and β which can easily be solved.

This was followed by considerable attention given to the solution of third degree (cubic) and fourth degree (quartic) equations by European mathematicians including Scipione dal Ferro (1465-1526), Niccolo Tartaglia (1500-1557), Girolamo Cardano, Ludovico Ferrari (1522-1565), Rafael Bombelli (1526-1572), Francois Viète and Ehrenfried von Tschirnhaus (1652-1708). In solving cubic and quartic equations, their work was a major milestone in the history of classical algebra. Like the quadratic formula (3.6.18), the roots of cubic and quartic equations can be expressed in terms of their coefficients. The systematic approach to finding a substitution that would eliminate both the linear and quadratic terms in the cubic equation, and the attempt to develop a similar rearrangement to reduce the quartic equation to root extractions led to the idea of a *resolvent*, an asymmetric function of the roots that assumes fewer values when the roots are permuted than there are roots. It was obvious that problems became more complicated as attempts were made to solve polynomial equations of even higher degrees. Up to this point, the classical approach had been algebraic as algebraic substitutions were used to reduce the equations to a simpler form. More precisely, the quadratic equation $ax^2 + bx + c = 0$ can be reduced to a simpler equation $z^2 = N$ by the substitution $z = x + \left(\frac{b}{2a}\right)$. Tschirnhaus discovered that a general cubic equation $x^3 + px + q = 0$ can be transformed into a simpler equation $z^3 = N$ by a substitution of the form $z = x^2 + rx + s$ for suitable constants r and s which can be found by solving only linear and quadratic equations. Similarly, a general quartic equation $ax^4 + bx^3 + cx^2 + dx + e = 0$ can be reduced to the equation $z^4 = pz^2 + qz + r$ by the substitution $x = z - (b/4a)$. Cardano and Ferrari had shown that the quartic equation can be solved by reducing it either to a linear and a cubic or two quadratic equations. In the meantime, complex numbers began to gain acceptance as possible roots of polynomial equations and attempts had been made to develop algebraic methods for solving all polynomial equations.

Cardano's treatment of cubic equation $x^3 + Ax^2 + Bx + C = 0$ can briefly be described as follows. The substitution of $x = y - \frac{1}{3}A$ reduces it to a simpler cubic equation in the form

$$y^3 + ay + b = 0. \tag{3.6.19}$$

Introducing $p = \sqrt[3]{-\frac{b}{2} + d}$, and $q = \sqrt[3]{-\frac{b}{2} - d}$, where $d = +\sqrt{\left(\frac{a}{3}\right)^3 + \left(\frac{b}{2}\right)^2}$, Cardano's solutions of three roots are given by $x_1 = p + q - \frac{A}{3}$, $(x_2, x_3) = -\frac{1}{2}(p+q) - \frac{A}{3} \pm i\frac{\sqrt{3}}{2}(p+q)$.

On the other hand, Viète also investigated both cubic and quartic equations, and simplified Cardano's treatment of cubic equation (3.6.19) by the substitution $y = \left(z - \frac{a}{3z}\right)$ to obtain the quadratic equation in z^3 as

$$z^3 - \frac{a^3}{27z^3} + b = 0. \tag{3.6.20}$$

The roots of this quadratic equation is given by

$$z^3 = -\frac{b}{2} \pm \sqrt{\frac{a^3}{27} + \frac{b^2}{4}} = N(\text{say}). \tag{3.6.21}$$

Viète's treatment of quartic equations was somewhat similar to that of Ferrari, but more direct that Ferrari's method. Viète's extraordinary trigonometrical skills helped him to deal with Cardano's *casus irreducibilis* of the cubic equation with three real roots. Putting $z = \cos\theta$ in the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ led to the cubic equation

$$z^3 - \frac{3}{4}\,z - \frac{1}{4}\cos 3\theta = 0. \tag{3.6.22}$$

Introducing $x = nz$, n is an arbitrary constant, in $x^3 - ax - b = 0$ gives

$$z^3 - \frac{a}{n^2}\,z - \frac{b}{n^3} = 0, \tag{3.6.23}$$

so that equating the coefficients of (3.6.22) and (3.6.23) gives $n = \sqrt{\frac{4a}{3}}$ and $\cos 3\theta = \left(4b/n^3\right) = \frac{1}{2}b/\sqrt{(a^3/27)}$. Viète obtained the values of $n = 2\sqrt{5}$ and $\cos 3\theta = \left(2/5\sqrt{5}\right)$ involved in a particular cubic equation $x^3 = 15x + 4$, and then determined three real roots $x_1 = n\cos\theta = 4$, $(x_1, x_2) = -2 \pm \sqrt{3}$.

Viète also derived the first exact formula for π using the trigonometric identity

$$\begin{aligned}\sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2^2\sin\left(\frac{\theta}{2^2}\right)\cos\left(\frac{\theta}{2^2}\right)\cos\left(\frac{\theta}{2}\right),\\ &= \cdots \quad \cdots \quad \cdots \quad \cdots,\\ &= 2^n\sin\left(\frac{\theta}{2^n}\right)\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2^2}\right)\ \cdots\ \cos\left(\frac{\theta}{2^n}\right),\end{aligned}$$

so that

$$\frac{\sin\theta}{\theta} = \left\{\frac{\sin(\theta/2^n)}{(\theta/2^n)}\right\}\left\{\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2^2}\right)\cdots\cos\left(\frac{\theta}{2^n}\right)\right\}. \tag{3.6.24}$$

In the limit as $n \to \infty$ with $\theta = \pi/2$, (3.6.24) reduces to

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\cdot\cos\frac{\pi}{8}\cdot\cos\frac{\pi}{16}\cdots. \tag{3.6.25}$$

Since $\cos\frac{\alpha}{2} = \sqrt{\frac{1}{2}(1+\cos\alpha)}$, (3.6.25) becomes

$$\begin{aligned}\frac{2}{\pi} &= p_1\,p_2\,p_3\ \cdots\\ &= \sqrt{\frac{1}{2}} \times \sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1}{2}}\right)} \times \sqrt{\frac{1}{2}\left(1+\sqrt{1+\sqrt{\frac{1}{2}}}\right)} \times \cdots,\end{aligned} \tag{3.6.26}$$

where $p_1 = \sqrt{\frac{1}{2}}$, and $p_{n+1} = \sqrt{\frac{1}{2}(1+p_n)}$.

Indeed, both Paolo Ruffini (1765-1822) in 1799 and Niels H. Abel (1802-1829) in 1824 clearly demonstrated that the general quintic equation *cannot* be solved by a single algebraic formula in terms of its coefficients. Abel proved a celebrated theorem on the non-solvability of the general quintic equation and the equations of higher degree by radicals. He also generalized the work of Gauss on the cyclotomic (circle-splitting) equation $x^n + x^{n-1} + \cdots + x + 1 = 0$ which led Gauss to the construction of the 17-sided regular polygon using a ruler and a compass. Abel proved that if every root of an polynomial equation can be generated by applying a given rational function successively to a single (premitive) root, such equation can be solved by radicals. Any two permutations that retain this function invariant must commute with each other. Any group whose elements commute is called an *Abelian group*. The investigation of solutions of algebraic equations was then completed by the breakthrough of the French mathematician Everisté Galois (1811-1832) who created a magnificent branch of mathematics known as *Abstract Algebra*, in particular, a part of it called the *Galois theory*, which expresses the solvability of an algebraic equation by radicals in terms of a group of permutations. The Galois theory includes as special cases results of Gauss' cyclotomic fields, $Q\left(\zeta = \exp\left(\frac{2\pi i}{p}\right)\right)$, where ζ is a primitive pth root of unity and p is an odd prime, and the construction of regular polygon as well as the Abel celebrated theorem on the non-solvability of the general polynomial equation of fifth and higher degrees by radicals. Since no algebraic formulas could be found to express the roots of a general quintic equation, a search continued for transcendental formula which were discovered by Charles Hermite (1822-1900) in 1858 and by Leopold Kronecker (1823-1891) in 1861. Such formulas involved elliptic integrals whose symmetric property had been investigated by Abel and others. Galois also mentioned some link between a algebraic equation and transcendental functions. Subsequently, Hermite successfully completed Galois' investigations in solving the quintic equation by means of the elliptic modular functions (see Dutta and Debnath (1965)). Camille Jordan (1838-1922) developed a general theory of groups of permutations governing the behavior of such transcendental functions.

Inspired by major contributions of his predecessors to the theory of equations, Gauss considered the polynomial equation

$$z^n - 1 = 0. \tag{3.6.27}$$

This is a algebraic version of the geometrical problem of constructing a regular polygon of n sides. He provided a remarkable proof to determine which regular polygons can be constructed by Euclidean methods (by a ruler and a compass) and which cannot. It turns out that a polygon of n sides can be constructed using a ruler and a compass if and only if $n = 2^m p_1 p_2 \cdots p_r$, where m is a non-negative integer, and p_r are distinct Fermat primes of the form $F_r = 2^{2^r} + 1$, $r = 0, 1, 2, \cdots$. Only five such primes F_r are known: 3, 5,17, 257,65, 537, corresponding to $r = 0, 1, 2, 3$ and 4. Gauss proved that a Euclidean solution of (3.6.27) is possible if and only if p is a Fermat prime. The equation (3.6.27) for $n = p$ is called *cyclotomic equation*. The set of complex solutions of this equation contains the number 1, and divides the unit circle into n equal parts. Before discussing Gauss' method of construction of the 17-sided ($F_2 = 17$) regular polygon, it is helpful to consider of $F_0 = 3$ which corresponds to an equilateral triangle that is easy to construct. The next simplest case of $F_1 = 5$ which corresponds to a regular pentagon. This is equivalent to the solution of the cyclotomic equation

$$z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1) = 0 \tag{3.6.28}$$

so that the real root is $z = 1$ and four other roots satisfy the equation

$$z^4 + z^3 + z^2 + z + 1 = z^2 + z + 1 + z^{-1} + z^{-2} = 0. \tag{3.6.29}$$

Introducing a new variable $w = z + z^{-1}$, (3.6.29) satisfies the quadratic equation

$$w^2 + w - 1 = 0. \tag{3.6.30}$$

Obviously, z can be obtained from the roots of (3.6.29) by solving the second quadratic equation, $w = z + z^{-1}$, that is,

$$z^2 - wz + 1 = 0. \tag{3.6.31}$$

Thus, equation (3.6.28) can be solved by extracting square roots only, and a regular pentagon can be constructed by using a ruler and a compass.

The case of $n = p = 17 = 2^4 + 1$ corresponds to the solution of the equation

$$z^{17} - 1 = (z-1)(z^{16} + z^{15} + \cdots + z + 1) = 0. \tag{3.6.32}$$

Writing $z^{17} - 1 = z^{17} - e^{2\pi ik} = 0$, where k is an integer so that the roots are

$$z_k = \exp\left(\frac{2\pi ik}{17}\right) = \cos\left(\frac{2\pi k}{17}\right) + i\sin\left(\frac{2\pi k}{17}\right), \tag{3.6.33}$$

where $k = 0, 1, 2, \cdots, 16$. It is noted that $z_0 = 1$, $z_k = z_1^k$ and $z_{17n+k} = z_k$ with integral n, and $z_{17-k} = z_k^{-1}$ for $k = 1, 2, \cdots, 16$.

The problem can be solved from the value of

$$z_1 + z_{16} = z_1 + \frac{1}{z_1} = 2\cos\left(\frac{2\pi}{17}\right), \tag{3.6.34}$$

where $\theta = \left(\frac{2\pi}{17}\right)$ represents the angle between any two consecutive sixteen points on the unit circle with center at the origin. Gauss discovered an ingenious method for organizing the 16 roots of (3.6.32) in a particular order, and then decompose the ordered sum into sums (called *periods*) containing 8, 4 and 2 respectively, and to do this in such a way that the values of the periods can be computed successively as the roots of quadratic equations. Gauss also provided a method of finding a *primitive root* of the associated congruence $a^s - 1 \equiv 0 \pmod{17}$. The number s is called a primitive root if the congruence has a solution for $s = 17 - 1 = 16$ but no smaller value of s. Thus, $a = 3$ is a primitive root and $3^{16} \equiv 1 \pmod{17}$. Without any further elaborate discussion of the remaining steps, we make reference to Appendix 5 of Hollingdale (1989) for details, and then state the side length, ℓ of the 17-sided regular polygon as

$$\frac{1}{8}\left[-1+\sqrt{17}+\sqrt{34-2\sqrt{17}} + \sqrt{68+12\sqrt{17}-16\sqrt{34+2\sqrt{17}}+2(-1+\sqrt{17})\sqrt{34-2\sqrt{17}}}\,\right]. \tag{3.6.35}$$

This seems to be a complicated expression, but it contains only square roots and no other irrational numbers. Thus, the 17-sided regular polygon can be constructed using a ruler and a compass only. This was indeed one of the monumental discoveries of Gauss when he was 19 years old, and this was the first (and only) advance on the problem of constructing a regular polygon since the Greek mathematics. Gauss was so proud of his discovery that he left an instruction to engrave a regular 17-sided polygon on his grave. Although his wish was never fulfilled, such a polygon was indeed inscribed on the side of the monument erected at his birthplace in Brunswick, Germany.

In order to develop a unified approach to finding solutions of polynomial equations of all degrees, Euler published a paper entitled 'De resolutione aequationum cuiusque gradus' (On the solution of equations of any degree) in 1762. In this paper, he proposed the solution of equation of nth degree in the form

$$x = \sqrt[n]{A_1} + \sqrt[n]{A_2} + \cdots + \sqrt[n]{A_{n-1}}, \tag{3.6.36}$$

where $A_1, A_2, \cdots, A_{n-1}$ are the roots of a resolvant equation of degree $(n-1)$. Thirty years later, he suggested an alternative formula of the same kind as

$$x = w + A\sqrt[n]{v} + B\sqrt[n]{v^2} + \cdots + Q\sqrt[n]{v^{n-1}}, \tag{3.6.37}$$

where w is real, v, and the coefficients $A, \cdots, Q$ can be determined by a method similar to Tschirnhaus' transformations. Such a form for the general solution of the quintic equation was employed by Abel in his proof which confirmed that no such solution could exist. Euler described a method of a general quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0, \tag{3.6.38}$$

which can be reduced by the transformation $y = x + \frac{a}{4}$ into the form

$$y^4 + py^2 + qy + r = 0. \tag{3.6.39}$$

The behavior of solutions of this quartic depends on the behavior of solutions of the resolvant cubic equation

$$z^3 + 2pz^2 + \left(p^2 - 4r\right) z - q^2 = 0. \tag{3.6.40}$$

Both Euler and d'Alembert made considerable progress in the understanding of general polynomial equations with methods of solutions. Euler observed that a quintic equation cannot be solved by algebraic methods. As stated earlier, both Abel and Galois provided a rigorous proof of this observation.

One of the major result in classical algebra is the celebrated *Fundamental Theorem of Algebra* which states that every nth degree algebraic equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \tag{3.6.41}$$

where a_n, a_{n-1}, $\cdots$, a_1, and a_0 are real numbers, has at least one real or complex root. This theorem was first formulated by A. Girard (1595-1632) in 1629 and then a rigorous formulation of this theorem was given by a great mathematician and philosopher, René Descartes in 1637, but its first proof was published by d'Alembert in 1746. In the same year, Euler presented his proof of this theorem at the Berlin Academy of Sciences. In his dissertation in 1799, Friedrich Gauss provided a first complete proof of the fundamental theorem of algebra based on the fact that the complex numbers are algebraically closed. Thus, Gauss proved the existence of the root of the equation (3.6.41), but he had doubt about the finding of an algebraic method of computing it from the coefficients of (3.6.41). It is important to note that the fundamental theorem of algebra is really an easy theorem to prove using the theory of complex functions of a complex variable, but a pure algebraic proof is a totally different matter.

In his celebrated work on number theory and algebra, Euler extensively investigated four major topics including the theory of congruences, algebraic numbers, Diophantine equations, and the law of quadratic reciprocity. Indeed, the law of quadratic reciprocity is perhaps one of the most original and fundamental discovery of the eighteenth century in number theory and algebra. This law is based on the ideas of congruences and quadratic residues which appeared in the works of Euler, Lagrange and Legendre. The symbol $a \equiv b \pmod{m}$ read as the number a is congruent to b modulo m means that $a-b$ is exactly divisible by m, where a, b, and m are integers. Then, b is called a *residue* of a modulo m.

The general *quadratic* (*second-order*) *congruence* with an odd prime modulus p is

$$ax^2 + bx + c \equiv 0 \qquad \pmod{p}, \tag{3.6.42}$$

where p does not divide a $(p \nmid a)$. If we write (3.6.42) as $4a\left(ax^2 + bx + c\right) \equiv 0 \pmod{p}$, or equivalently as $(2ax+b)^2 + \left(4ac - b^2\right) \equiv 0 \pmod{p}$. This is equivalent to the system

$$\left.\begin{aligned} X^2 &\equiv A \qquad \pmod{p} \\ 2ax + b &\equiv X \qquad \pmod{p} \end{aligned}\right\} \tag{3.6.43ab}$$

where $A \equiv b^2 - 4ac \pmod{p}$. Since $p \nmid A$, the congruence (3.6.43b) has a unique solution $x \equiv x_0 \pmod{p}$, and hence, the solution of the original quadratic congruence (3.6.42) reduces to that solving (3.6.43a).

In the language introduced by Euler in 1754 and then adopted by Gauss, if p is prime and a is an integer such that $p \nmid a$, and if the quadratic congruence $x^2 \equiv a \pmod{p}$ has a solution, then a is called a *quadratic residue of prime p*. If no solution exists, then a is called a *quadratic nonresidue of p*. For example, if $a = 4$ and $p = 5$, the congruence is solvable with solutions are 2, 3, 7, $\cdots$, but if $a = 3$, it is *not*.

It is important to observe that algebraic results for polynomial congruences are somewhat similar to those of polynomial equations if the moduli are prime numbers. This leads to the generalization of the Fundamental Theorem of Algebra to an arbitrary field $\mathbf{F}$ due to Lagrange. If $\mathbf{F}[x]$ is the set of all polynomials in x with coefficients in $\mathbf{F}$, then a nonzero polynomial $f(x) \in \mathbf{F}[x]$ of degree n can have at most n zeros in $\mathbf{F}$. As a corollary, we can state that if a polynomial $f(x)$ of degree n, is not identically congruent to 0 modulo prime p, then $f(x) \equiv 0 \pmod{p}$ has *at most n* roots.

In his book entitled *Essai sur la Théorie des nombres* published in 1798, a French mathematician Adrian-Marie Legendre (1752-1833) introduced the symbol that is now known as the *Legendre symbol*, $\left(\frac{a}{p}\right)$:

$$\left(\frac{a}{p}\right) = \begin{cases} +1 \text{ if } & a \text{ is a quadratic residue modulo } p > 2, \\ \\ 0 \text{ if } & p \mid a, \\ \\ -1 \text{ if } & a \text{ is a quadratic nonresidue modulo } p > 2. \end{cases}$$

The Legendre symbol is not defined for $p = 2$. Because of the multiplication rule for residues, it turns out

$$\left(\frac{a}{p}\right) \cdot \left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right). \tag{3.6.44}$$

However, Euler developed a criterion for a number a to be a quadratic residues without using the Legendre symbol. More importantly, in his works published in 1751 and 1783, Euler conjectured what is now known as the *law of quadratic reciprocity*. This law expresses an elegant and reciprocal relationship between the pair of congruences:

$$x^2 \equiv p \pmod{q} \qquad \text{and} \qquad x^2 \equiv q \pmod{p} \tag{3.6.45ab}$$

where both p and q are odd primes.

In order to give an equivalent statement of this law, we introduce an integer $n = \frac{1}{2}(p-1) \cdot \frac{1}{2}(q-1)$. The law states that if n is odd, then one and only one of the congruences (3.6.45ab) is solvable, and if n is even, then either both or neither of the congruences are solvable. For example, if $p = 7$ and $q = 13$ so that $n = 18$, then both congruences (3.6.45ab) are solvable. On the other hand, if $p = 5$ and $q = 13$ giving $n = 12$, the law states that either both or neither congruences (3.6.45ab) are solvable. In fact, neither is solvable. If $p = 3$ and $q = 7$, then $n = 3$. Thus, (3.6.45b) is solvable, but (3.6.45a) is not solvable. Examples including $p = 3$, $q = 11$; $p = 7$, $q = 5$; and $p = 5$, $q = 11$ are left for the reader as exercises.

In symbolic form, Euler's law of quadratic reciprocity can elegantly be stated as

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^n, \tag{3.6.46}$$

where $n = \frac{1}{4}(p-1)(q-1)$. This means that if the exponent n of (-1) is even, p is a quadratic residue of q and vice-versa, or neither is a quadratic residue of the other. When the exponent is odd, which occurs when p and q

are of the form $4k+3$, one prime would be a quadratic residue of the other, but not the second of the first. On the other hand, if at least one of p, q is a $(4k+3)$-prime, then at least one of $\frac{1}{2}(p-1)$, $\frac{1}{2}(q-1)$ is even, and so, $(-1)^n = +1$. This implies that the two Legendre symbols are either both $+1$ or both -1. Of particular interest are the following special cases:

$$\left(-\frac{1}{p}\right) = (-1)^{\frac{1}{2}(p-1)}, \tag{3.6.47}$$

which was found, in a general form, as the Euler criterion

$$\left(\frac{2}{p}\right) = (-1)^{\frac{1}{8}(p^2-1)}, \tag{3.6.48}$$

that is, 2 is a quadratic residue if and only if $p \equiv \pm 1 \pmod 8$.

Historically, the law of quadratic reciprocity was discovered empirically and independently by Euler in 1722 and Legendre in 1785. In 1796, Gauss not only stated this law in elegant form, but gave several complete proofs, and hence, it is now known as the *Gauss quadratic reciprocity law*. In order to extend the law to higher powers, Gauss discovered the *Gaussian integers*, that is, complex numbers of the form $m + n\sqrt{-1}$. Gauss proved that the fundamental ideas of prime and composite number make sense in this context just as in the ordinary integers and that every such number has a unique representation up to multiplication by ± 1 and $\pm\sqrt{-1}$ as a product of irreducible factors. In fact, a prime number of the form $4k+1$ *cannot* be prime in this context, as it is a sum of two squares: $4k+1 = p^2 + q^2 = (p+q\sqrt{-1})\,(p-q\sqrt{-1})$. This kind of generalization of the concept of prime number to the Gaussian integers is an early classic example of numerous generalization and abstraction in modern mathematics.

The Euler criterion can be used to test quadratic residue. More precisely, the integer a, $(a,p) = 1$, is a quadratic residue of modulo odd p, that is, $\left(\frac{a}{p}\right) = 1$, if and only if

$$a^{\frac{1}{2}(p-1)} \equiv 1 \pmod{p}, \tag{3.6.49}$$

it is a quadratic nonresidue if and only if

$$a^{\frac{1}{2}(p-1)} \equiv -1 \pmod{p}. \tag{3.6.50}$$

Euler's formulation of quadratic reciprocity law is a special case of Emil Artin's (1896-1962) general reciprocity law for quadratic number fields, and Peter Gustav Dirichlet's (1804-1851) quadratic reciprocity law

$$\zeta_K(s) = \zeta(s)L(s,\chi), \tag{3.6.51}$$

where $\zeta_K(s)$ is Richard Dedekind's (1831-1916) zeta function of a number field K, $\zeta(s)$ is the Riemann zeta function and $L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$ is the Dirichlet L-function attached to a quadratic character $\chi \mod q$.

David Hilbert generalized the quadratic reciprocity law to algebraic number fields in terms of the *Hilbert symbol* in the form

$$\prod_p \left(\frac{a,b}{p}\right) = 1, \tag{3.6.52}$$

where the product is over all primes including $p = \infty$ and for a, $b \in Q_p$, the Hilbert symbol is defined by

$$\left(\frac{a,b}{p}\right) = \begin{cases} +1 & \text{if } ax^2 + by^2 - z^2 \text{ has a solution } (x,y,z) \in Q_p^3, \\ -1 & \text{otherwise.} \end{cases}$$

Hilbert also proved that if p is an odd prime and a is not divisible by p, his symbol satisfies

$$\left(\frac{a,b}{p}\right) = \left(\frac{a}{p}\right)^{\nu_p(b)}, \tag{3.6.53}$$

where $\nu_p(b)$ is the exponent to which p appears in the prime factorization of b. If a, b are two distinct primes p, q, the Euler law (3.6.46) follows from Hilbert's law (3.6.52) together with (3.6.53).

In the case of $p = \infty$, $\left(\frac{a,b}{\infty}\right) = 1$ if and only if a and b are *not* both negative, that is, if the equation $ax^2 + by^2 - z^2 = 0$ has solutions in real number for fixed a and b. Thus, the Hilbert product makes sense in view of the fact that $\left(\frac{a,b}{p}\right) = 1$ for fixed a and b and for all but finitely many primes p. However, problems arise only when multiplication by a or b leads to many of these quadratic residues. For example, if a and b are positive primes, then only these two primes contribute to the product so that the two resulting factors can be associated with $\left(\frac{a}{b}\right)$ and $\left(\frac{b}{a}\right)$ which leads to the quadratic reciprocity law.

Finally, around 1770, Euler introduced an *Euler Brick* which is simply a rectangular box (or *cuboid*) in which all of three sides a, b, c have integer values and in which all three diagonals

$$d_{a,b} = \sqrt{a^2+b^2},\ d_{b,c} = \sqrt{b^2+c^2} \text{ and } d_{c,a} = \sqrt{c^2+a^2} \tag{3.6.54}$$

are also integers as shown in Figure 3.4, where $a > b > c$. The smallest Euler brick has sides $(a,b,c) = (240,117,44)$ with $d_{a,b} = 267$, $d_{b,c} = 125$ and $d_{c,a} = 244$. Other Euler's bricks are found including $(275,252,240)$, (693, 480, 140), (720, 132, 85) and (792, 231, 160) which are listed by Guy (1994). Kraitchik obtained 257 Euler's bricks with the odd side less than 1 million, whereas Helenius has compiled a list of the 5003 smallest (measured by the longest side) Euler bricks. In 1740, N. Saunderson obtained a parametric solution for some Euler bricks (but *not all possible Euler bricks*). In 1770 and 1772, Euler discovered two parametric solutions which did not produce all possible solutions. If (a', b', c') is a Pythogorean triple, then

$$(a, b, c) = \left[a'(4b'^2 - c'^2), b'(4a'^2 - c'^2), 4a'b'c'\right] \tag{3.6.55}$$

is a Euler brick with face diagonals $d_{a,b} = c'^3$, $d_{b,c} = b(4a'^2 + c'^2)$ and $d_{c,a} = a'(4b'^2 + c'^2)$ (see Dickson (2005)). If (a, b, c) is a given Euler brick, then (bc, ca, ab) is also an Euler brick.

A *perfect Euler brick* (or a *perfect cuboid*) of sides a, b, c $(a > b > c)$ is an Euler brick in which the length $\sqrt{a^2 + b^2 + c^2}$ of the space diagonal is also an integer. It is not yet known whether a perfect Euler brick exists as there is no example of a perfect cuboid found. A major question relating to the existence of perfect cuboid's remains unsolved. However, an extensive computer search for the integer cuboids revealed that the smallest side of a perfect cuboid is at least 4.3 billion. A lot of recent computations also revealed that two of the three face diagonals and space diagonal are integers.

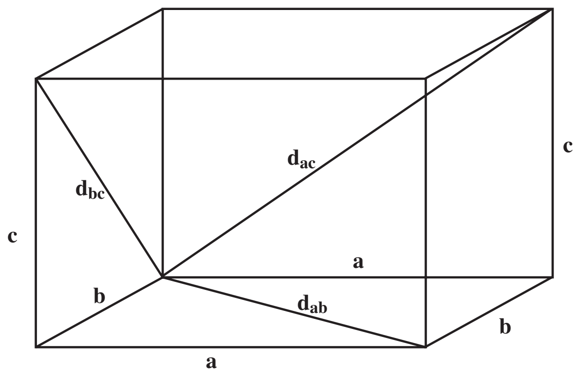


Fig. 3.4 Euler's brick.

Chapter 4

Euler's Contributions to Geometry and Spherical Trigonometry

> "Although the Greeks worked fruitfully, not only in geometry but also in the most varied fields of mathematics, nevertheless we today have gone beyond them everywhere and certainly also in geometry."

Fleix Klein

> "Geometry is useful not only in algebra, analysis, and cosmology, but also in kinematics and crystallography (where it is associated with the theory of groups), in statistics (where finite geometries help in the design of experiments), and even in botany."

H. S. M. Coxeter

4.1 Introduction

Euler was fully inspired by the Euclid monumental work *The Elements* which is by far the most influential work in geometry, classical algebra, number theory and the remarkable model for the axiomatic method in mathematics. He was motivated by René Descartes' book *Discours de la Méthod* (Lecture on the method) in which Descartes founded analytic geometry as a synthesis of geometry and algebra. This marked the beginning of the golden age of modern mathematics.

Throughout this chapter, we describe a triangle ABC by its three angles A, B, C, and three sides $BC = a$, $CA = b$ and $AB = c$ so that the sum of three angles, $(A + B + C) = 180^\circ = \pi$. We denote the perimeter of the triangle ABC by $2s = (a + b + c)$ and the area by $\triangle ABC$, or simply by $\triangle$.

4.2 Euler's Work in Plane Geometry

In modern plane geometry, the fundamental concept of length (or distance) of a line segment was introduced with a magnitude as well as a sign. If A and B are two distinct points on a straight line, as shown in Figure 4.1 (a), the directed line segments AB and BA are equal in magnitude, but opposite in sign. Obviously, the line segments AB and BA are expressed by the equation

$$AB = -BA, \qquad AB = OB - OA. \tag{4.2.1}$$

Or, equivalently, the fundamental formulas are

$$AB + BA = 0, \qquad AB = AO + OB. \tag{4.2.2}$$

So, $AO + OB$ implies that the point travels from A to O and then from O to B, and hence, the point is moved from A to B.

If A, B and C are distinct points on a straight line, as shown in Figure 4.1 (b), then

$$AB + BC + CA = 0. \tag{4.2.3}$$

Euler's Theorem 4.2.1. If A, B, C and D are distinct points on a straight line, as shown in Figure 4.1 (b), then the following identity holds among the line segments determined by these points

$$AB \cdot CD + AC \cdot DB + BC \cdot AD = 0. \tag{4.2.4}$$

The proof of this theorem follows from the following representation of the left hand side of (4.2.4) as

$$(AD - BD) \cdot CD + (AD - CD) \cdot DB + (BD - CD) \cdot AD. \tag{4.2.5}$$

Or,

$$(AD + DB) \cdot CD + (AD - CD) \cdot DB - (DB + CD) \cdot AD \tag{4.2.6}$$

which vanishes identically. Thus, identity (4.2.4) is proved.

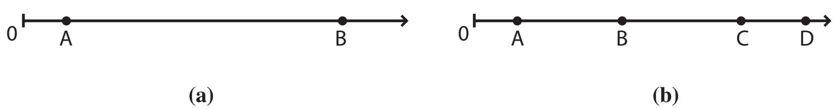


Fig. 4.1 (a) Directed line segments AB and BA, (b) Directed line segments AB, BC and CA.

In 1765, Euler first made an attempt to generalize the famous theorem which states that the angles at the base of an isosceles triangle are equal and conversely. He formulated that the ratio of the angles $\angle A : \angle B = m : n$ of a triangle CAB for $n = 1$. If $m = n = 1$, $\angle A = \angle B$, then $a = b$, so the triangle CAB is an isosceles triangle. He proved that if $\angle B : \angle A = 2 : 1$, then $b^2 - a^2 = ac$, and conversely.

If $\angle B = 3\angle A$, then he proved that $a(b^2 - a^2 + c^2)^2 = b^2c^2(a + b)$ and conversely. He continued to prove results up to $m = 13$.

In 1678, an Italian mathematician, Giovanni Ceva (1647-1736) published a treatise on *Geometry* which included many basic properties of triangles. We state here a couple of Ceva's theorem without proofs as they are elementary exercises. In Figure 4.2, the points L, M, N are called *feet* of the *Cevians* AL, BM and CN respectively, the point P is called the *Ceva point* and the triangle LMN is called the *pedal triangle* of the Ceva point P.

Ceva's Theorem 4.2.2. If three Cevians AL, BM and CN are drawn from the vertices of a triangle ABC so that they meet at P, they divide the opposite sides into six line segments such that the product of the three segments with no common ends is equal to the product of the other three line segments. More precisely, as shown in Figure 4.2,

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1. \tag{4.2.7}$$

The converse of this theorem is also true.

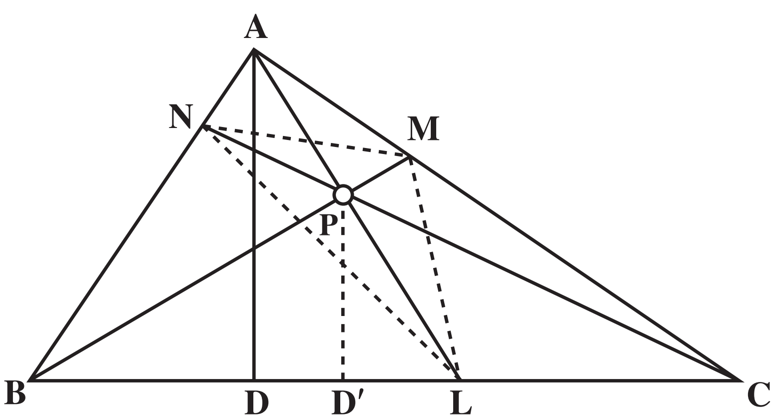


Fig. 4.2 A triangle ABC, and the pedal triangle LMN.

Ceva's Theorem 4.2.3. If LMN is a pedal triangle of the Ceva point P for the triangle ABC, then

$$\frac{PL}{AL} + \frac{PM}{BM} + \frac{PN}{CN} = 1. \tag{4.2.8}$$

The proof of this theorem follows from similar triangles BPC and BAC.

It seems that Euler was also inspired by Ceva's treatise on Geometry and proved several interesting theorems.

Euler's Theorem 4.2.4. In any triangle ABC, AD, BE, and CF are three Cevians meeting at the Ceva point O, then

$$\frac{OD}{AD} + \frac{OE}{BE} + \frac{OF}{CF} = 1. \tag{4.2.9}$$

The proof of this theorem follows from similar triangles BPC and BAC.

Proof. In Figure 4.3, we draw two line segments OP and OQ parallel to AB and AC respectively so that

$$BP + PQ + QC = BC.$$

Or,

$$\frac{BP}{BC} + \frac{PQ}{BC} + \frac{QC}{BC} = 1. \tag{4.2.10}$$

Obviously, there are three pairs of similar triangles. Since triangles BEC and BOQ are similar, so,

$$\frac{QC}{BC} = \frac{OE}{BE}. \tag{4.2.11}$$

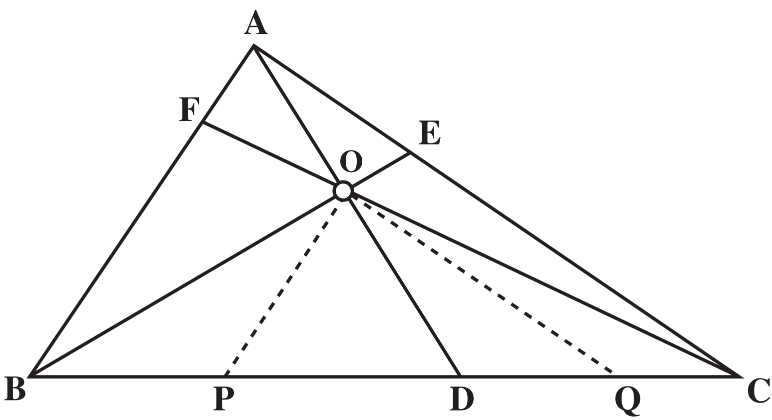


Fig. 4.3 ABC is a triangle with the three Cevians AD, BE and CF.

Similar, it follows from similar triangles CFB and COP that

$$\frac{BP}{BC} = \frac{OF}{CF}, \tag{4.2.12}$$

and from similar triangles POQ and ABC that

$$\frac{PQ}{BC} = \frac{OD}{AD}. \tag{4.2.13}$$

Substituting, (4.2.11)–(4.2.13) into (4.2.10) yields (4.2.9). The proof is complete.

We next introduce three real numbers α, β, γ so that $AO = \alpha \cdot OD$, $BO = \beta \cdot OE$ and $CO = \gamma \cdot OF$ so that

$$\alpha + 1 = \frac{AD}{OD}, \quad \beta + 1 = \frac{BE}{OE}, \quad \gamma + 1 = \frac{CF}{OF}. \tag{4.2.14}$$

Consequently, (4.2.9) reduces to the form

$$\frac{1}{\alpha+1} + \frac{1}{\beta+1} + \frac{1}{\gamma+1} = 1. \tag{4.2.15}$$

This is equivalent to

$$\frac{\alpha}{\alpha+1} + \frac{\beta}{\beta+1} + \frac{\gamma}{\gamma+1} = 2 \tag{4.2.16}$$

which is, after simple calculation, equivalent to

$$\alpha\beta\gamma = (\alpha + \beta + \gamma) + 2. \tag{4.2.17}$$

Or, equivalently,

$$\frac{AO}{OD} \cdot \frac{BO}{OE} \cdot \frac{CO}{OF} = \frac{AO}{OD} + \frac{BO}{OE} + \frac{CO}{OF} + 2. \tag{4.2.18}$$

This is true for any triangle ABC, as shown in Figure 4.3.

Theorem 4.2.5. *Euler Law of Quadrilateral.* If $ABCD$ is a quadrilateral with diagonals AC and BD, and if a parallalogram $ABCE$ is constructed with two sides BC, AE, AB and CE, and if points D and E are joined to form the lines DE, then

$$AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2 + DE^2. \tag{4.2.19}$$

Proof. Since $ABCE$ is a parallalogram with $AE = BC$ and diagonals AC and BE, we consider a point F as shown in Figure 4.4 so that CF is parallel to AD and BF is parallel to DE. Clearly, two triangles BCF and ADE are congruent to each other.

We then draw lines AF, DF and EF and consider the two parallalograms $ADCF$ and $BDEF$ with diagonals AC, DF and BE, DF respectively. Since angles $\angle ADC$ and $\angle DCF$ are supplementary, so

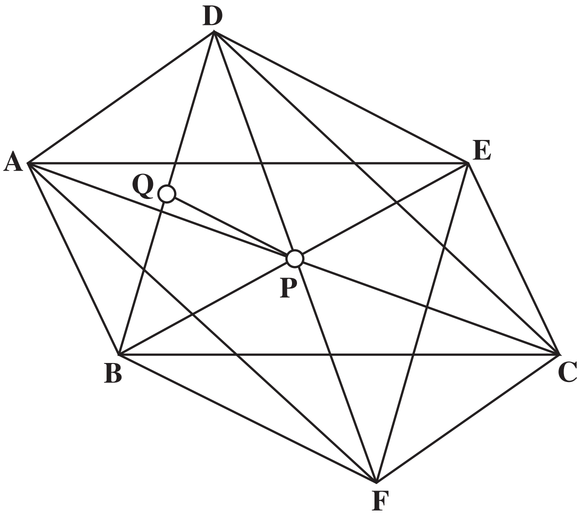


Fig. 4.4 A Quadrilateral $ABCD$.

$\cos(\angle DCF) = \cos(\pi - \angle ADC) = -\cos(\angle ADC)$. We apply the cosine law to the two triangles ADC and DCF to obtain

$$AC^2 = AD^2 + DC^2 - 2AD \cdot DC \cos(\angle ADC), \tag{4.2.20}$$

$$DF^2 = DC^2 + CF^2 + 2DC \cdot CF \cos(\angle ADC). \tag{4.2.21}$$

Adding (4.2.20) and (4.2.21) gives

$$AC^2 + DF^2 = 2\left(AD^2 + DC^2\right). \tag{4.2.22}$$

Similarly, for the parallalogram $BDEF$,

$$BE^2 + DF^2 = 2\left(BD^2 + DE^2\right). \tag{4.2.23}$$

Equating the value of DF^2 from (4.2.22) and (4.2.23) gives

$$2\left(AD^2 + DC^2\right) = 2\left(BD^2 + DE^2\right) + AC^2 - BE^2. \tag{4.2.24}$$

Similarly, it follows from the parallalogram $ABCE$, that

$$2\left(AB^2 + BC^2\right) = AC^2 + BE^2. \tag{4.2.25}$$

Adding (4.2.24) and (4.2.25) yields the *Euler law of quadrilateral* as

$$\left(AB^2 + BC^2 + CD^2 + AD^2\right) = \left(AC^2 + BD^2 + DE^2\right). \tag{4.2.26}$$

The following corollaries are in order:

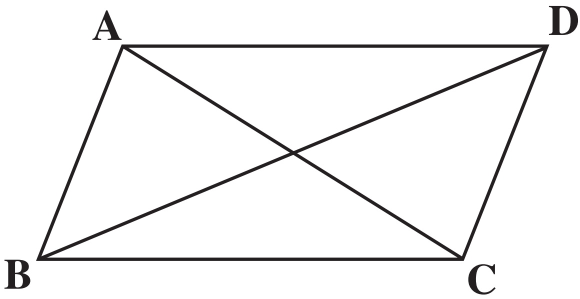


Fig. 4.5 A Parallalogram $ABCD$.

Corollary 4.2.1. For the quadrilateral $ABCD$,

$$AB^2 + BC^2 + CD^2 + AD^2 \geq AC^2 + BD^2, \tag{4.2.27}$$

where equality holds when the quadrilateral $ABCD$ is a parallalogram as shown in Figure 4.5 so that $DE \equiv 0$ so that

$$AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2. \tag{4.2.28}$$

That is, the sum of the squares of the sides of the parallalogram $ABCD$ is equal to the sum of the squares of the diagonals.

Corollary 4.2.2. In Figure 4.4, if the line PQ is joined by the mid-points P and Q of the diagonals AC and BD respectively, then

$$PQ = \frac{1}{2} DE. \tag{4.2.29}$$

In a quadrilateral $ABCD$, if P and Q are the midpoints of the diagonals AC and BD respectively, then

$$AB^2 + BC^2 + CD^2 + AD^2 = AC^2 + BD^2 + 4PQ^2. \tag{4.2.30}$$

This readily follows from (4.2.26).

Finally, another corollary which is the famous *Euler–Pythagoras theorem* follows from the Euler law (4.2.30) of quadrilateral:

Corollary 4.2.3. (The Euler–Pythagoras Theorem). If $ABCD$ is a rectangle in Figure 4.6 (b), then $P = Q$, $PQ = 0$, $AC = BD$, $CD = AB$ and $AD = BC$. The identity (4.2.30) reduces to

$$AB^2 + BC^2 = AC^2. \tag{4.2.31}$$

This is the famous *Euler–Pythagoras theorem* for the right angled triangle ABC.

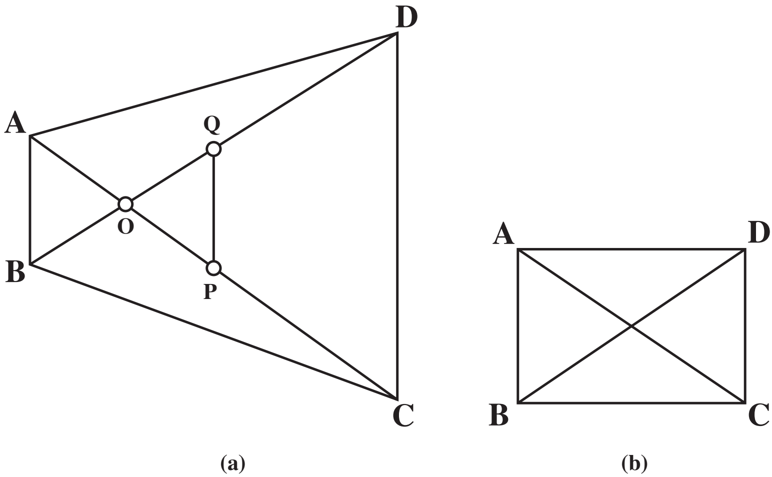


Fig. 4.6 (a) A Quadrilateral $ABCD$, (b) A Rectangle $ABCD$.

In about 150 A.D., the famous Greek astronomer, Claudius Ptolemy proved the following celebrated theorem for a cyclic quadrilateral.

Ptolemy's Theorem 4.2.6. In a cyclic quadrilateral, the product of the two diagonals is equal to the sum of the products of the two pairs of opposite sides, and conversely.

Proof. In Figure 4.7, $ABCD$ is a cyclic quadrilateral with sides $AB = a$, $BC = b$, $CD = c$, $DA = d$ and diagonals $AC = x$ and $BD = y$.

We draw an angle $\angle CDE = \angle ADB$ so that the two triangles ADB and CDE become similar as they have equal angles $\angle ABD$ and $\angle DCE$. Consequently,

$$\frac{a}{CE} = \frac{y}{c}.$$

Or,

$$CE \cdot y = ac. \tag{4.2.32}$$

Similarly, it follows from the similar triangles BDC and ADE that

$$\frac{b}{EA} = \frac{y}{d}.$$

Or,

$$EA \cdot y = bd. \tag{4.2.33}$$

Adding (4.2.32) and (4.2.33) gives

$$y(CE + EA) = (ac + bd).$$

Since $CE + EA = AC = x$, hence,

$$xy = ac + bd. \tag{4.2.34}$$

This is the assertion of Theorem 4.2.6. Conversely, if (4.2.34) holds for a quadrilateral $ABCD$, then $ABCD$ is cyclic.

In particular, if a cyclic quadrilateral $ABCD$ is a cyclic rectangle $ABCD$, then $a = c$, $b = d$ and $x = y$ so that Ptolemy's theorem reduces to the Euler–Pythagoras theorem

$$x^2 = a^2 + b^2. \tag{4.2.35}$$

Stewart's Theorem 4.2.7. If a, b, c are three sides of a triangle and if L is any point drawn from the vertex A on the side BC as shown in Figure 4.2, so that $BL = p$ and $LC = q$, then the length $x_a = AL$ is given by the quadratic equation

$$ax_a^2 = pb^2 + qc^2 - apq. \tag{4.2.36}$$

Proof. Using the law of cosines to the triangles ABL and ALC, as shown in Figure 4.2, we obtain

$$c^2 = x_a^2 + p^2 - 2x_a p \cos ALB, \tag{4.2.37}$$

$$b^2 = x_a^2 + q^2 + 2x_a q \cos ALB. \tag{4.2.38}$$

Multiplying (4.2.37) by q and (4.2.38) by p and adding these results with the fact that $p + q = a$ yields the desired result (4.2.36).

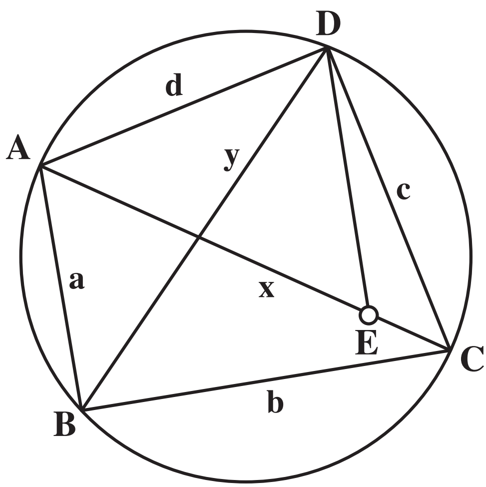


Fig. 4.7 A cyclic quadrilateral $ABCD$.

Remark. This theorem can be used to determine the lengths of three cevians AL, BM and CN. In particular, the lengths of the medians and the bisectors of the angles of a triangle can be found by means of Stewart's theorem which was proved by Matthew Stewart (1717–1785) in 1746, who succeeded Colin Maclaurin as professor of mathematics at the University of Edinburgh. If $p = q = \frac{a}{2}$, then (4.2.36) gives the length x_a of the median AL as

$$x_a^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}a^2. \tag{4.2.39}$$

Using the cyclic permulation of letters leads to corresponding formulas of the two other medians

$$x_b^2 = \frac{1}{2}(c^2 + a^2) - \frac{1}{4}b^2, \quad x_c^2 = \frac{1}{2}(a^2 + b^2) - \frac{1}{4}c^2. \tag{4.2.40}$$

Adding results (4.2.39) and (4.2.40) gives

$$x_a^2 + x_b^2 + x_c^2 = \frac{3}{4}(a^2 + b^2 + c^2). \tag{4.2.41}$$

This means that the sum of the squares of the three medians of a triangle is equal to three-fourths the sum of the squares of sides.

On the other hand, if the cevian AL is the bisector of the angle A, we then find that $\frac{p}{q} = \frac{c}{b}$ so that $\frac{p}{a} = \frac{c}{b+c}$, and $\frac{q}{a} = \frac{b}{b+c}$. Substituting the values p and q in (4.2.36) yields

$$x_a^2 = bc\left[1 - \frac{a^2}{(b+c)^2}\right]. \tag{4.2.42}$$

Similar expressions for x_b^2 and x_c^2 can be obtained by cyclic permulation of letters.

Finally, there is a remarkable formula discovered by Heron of Alexandria (10-70 A.D.) in about 60 A.D. If $\triangle$ denotes the area of a triangle ABC in terms of its side lengths;

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}, \tag{4.2.43}$$

where $2s = a + b + c$ is the perimeter of the triangle ABC.

Making reference of Figure 4.2, the area of the triangle ABC is given by

$$\triangle = \frac{1}{2}BC \cdot AD = \frac{1}{2}ah = \frac{1}{2}ab\sin C. \tag{4.2.44}$$

Similarly, two more formulas for the area of ABC are obtained by cyclic permutation of the quantities in (4.2.44):

$$\triangle = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B. \tag{4.2.45}$$

It then follows from (4.2.44) that

$$\sin^2 C = \frac{4\triangle^2}{a^2b^2}, \tag{4.2.46}$$

and from the law of cosines that

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

so that

$$\cos^2 C = \frac{(a^2+b^2-c^2)^2}{4a^2b^2}. \tag{4.2.47}$$

Adding (4.2.46) and (4.2.47) gives

$$\sin^2 C + \cos^2 C = \frac{4\triangle^2}{a^2b^2} + \frac{(a^2+b^2-c^2)^2}{4a^2b^2} = 1.$$

Or,

$$16\triangle^2 + (a^2+b^2-c^2)^2 = 4a^2b^2. \tag{4.2.48}$$

Or,

$$\begin{aligned} 16\triangle^2 &= 4a^2b^2 - (a^2+b^2-c^2)^2 \\ &= \left\{2ab + (a^2+b^2-c^2)\right\}\left\{2ab - (a^2+b^2-c^2)\right\} \\ &= \left\{(a+b)^2 - c^2\right\}\left\{c^2 - (a-b)^2\right\}. \end{aligned}$$

Therefore,

$$\triangle = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}. \tag{4.2.49}$$

Using $2s = a+b+c$, $a+b-c = 2(s-c)$, $b+c-a = 2(s-a)$ and $c+a-b = 2(s-b)$, the area $\triangle$ reduces to the form

$$\triangle = \sqrt{s(s-a)(s-b)(s-c)}. \tag{4.2.50}$$

This is the *celebrated Heron's formula* for the area of a triangle ABC.

Remark. Heron's formula seems strange in the sense that there is a square root of the product of four quantities. In almost all formulas for an area, the area is usually expressed by a product of two quantities.

4.3 Incircle, Incenter and Heron's Formula for an Area of a Triangle

The *incenter* is the center of the *inscribed circle* (or *incircle*) that touches three sides BC, CA and AB of the triangle ABC at P, Q and R as in Figure 4.8. We denote the incenter by I and inradius by r.

Clearly, all three bisectors of the three angles of a triangle ABC meet at the incenter I of the inscribed circle so that the radius IP of the incircle is perpendicular to the sides of the triangle ABC.

It follows from Figure 4.8 that

$$\begin{aligned}\triangle ABC &= \triangle BIC + \triangle CIA + \triangle AIB \\ &= \frac{1}{2}\,ar + \frac{1}{2}\,br + \frac{1}{2}\,cr = \frac{1}{2}(a+b+c)\cdot r = rs. \qquad (4.3.1)\end{aligned}$$

We denote $BP = x$, $PC = y$ so that $x + y = a$. Similarly, $CQ = y$ and $QA = z$ so that $y + z = b$ and $AR = z$ and $RB = x$, then $z + x = c$. Thus, $2s = (a + b + c) = 2(x + y + z)$. Evidently, $s - a = x + y + z - (x + y) = z$, $s - b = x$ and $s - c = y$. And $\triangle ABC = r(x + y + z)$. The use of the arithematic-geometric mean inequalities $x + y \geq 2\sqrt{xy}$, $y + z \geq 2\sqrt{yz}$ and $z + x \geq 2\sqrt{zx}$ gives the *Lehmus* or *Padoa inequality*:

$$(x+y)(y+z)(z+x) \geq 8xyz = (2x)(2y)(2z).$$

Or,

$$abc \geq (a+b-c)(b+c-a)(c+a-b).$$

Also,

$$a = x + y = BP + PC = r\left(\cot\frac{1}{2}B + \cot\frac{1}{2}C\right), \qquad (4.3.2)$$

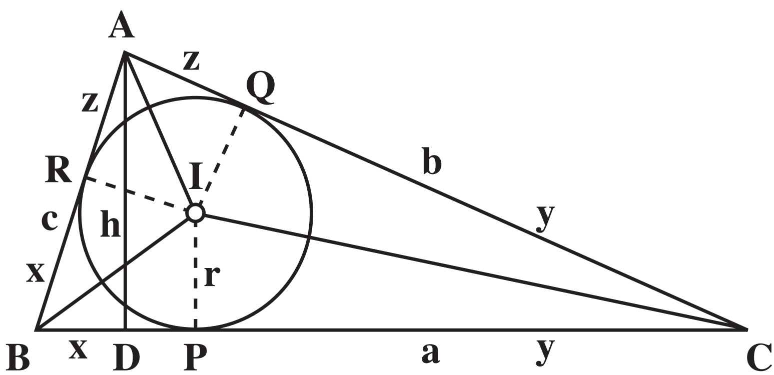


Fig. 4.8 Incenter and Incircle.

whence

$$r = a \sin \frac{1}{2} B \, \sin \frac{1}{2} C \, \sec \frac{1}{2} A. \tag{4.3.3}$$

It follows from Figure 4.8 that the altitude $AD = h_a$ is given by

$$\begin{aligned} h_a = AD = c \sin B = 2c \, \sin \frac{B}{2} \cos \frac{B}{2} \\ = 2c \cdot \frac{r}{\sqrt{r^2 + x^2}} \cdot \frac{x}{\sqrt{r^2 + x^2}} = \frac{2(x+z)rx}{(r^2 + x^2)}. \end{aligned} \tag{4.3.4}$$

Thus, the area of the triangle ABC is

$$\triangle ABC = \frac{1}{2} a \, h_a = \frac{1}{2}(x+y) \cdot \frac{2(x+z)rx}{(r^2+x^2)} = \frac{rx(x+y)(x+z)}{r^2 + x^2}. \tag{4.3.5}$$

In view of (4.3.1) and (4.3.5), it follows that

$$s(r^2 + x^2) = x(x+y)(x+z) = x(sx + yz).$$

Or,

$$sr^2 = xyz. \tag{4.3.6}$$

Thus, the area of the triangle ABC is

$$\triangle = rs = \sqrt{s \cdot sr^2} = \sqrt{sxyz} = \sqrt{s(s-a)(s-b)(s-c)}. \tag{4.3.7}$$

This is again *Heron's formula* for the area of a triangle.

It can be shown that the length ℓ_A of the bisector, AI of the angle A in Figure 4.8 is

$$\ell_A = \left(\frac{2bc}{b+c} \right) \cos \frac{A}{2} = \frac{\sqrt{bc\{(b+c)^2 - a^2 y\}}}{bc}. \tag{4.3.8}$$

It seems that one of the great Hindu mathematicians, Brahmagupta in the seventh century knew the Ptolemy's Theorem 4.2.6 on the cyclic quadrilateral, and discovered a remarkable theorem:

Brahmagupta's Theorem 4.3.1. The area $\square$ of a cyclic quadrilateral of $ABCD$ of sides a, b, c and d is given by

$$\square = \sqrt{(s-a)(s-b)(s-c)(s-d)}, \tag{4.3.9}$$

where $2s = (a+b+c+d)$ is the perimeter of the quadrilateral $ABCD$. Obviously, Heron's formula (4.3.7) for the area of a triangle is a special case of (4.3.9).

It is interesting to point out that the square of the area $\square$ of any arbitrary quadrilateral $ABCD$ (see Figure 4.7) is given by the following formula

$$\square^2 = (s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \left(\frac{A+C}{2} \right), \tag{4.3.10}$$

when A and C are the opposite vertex angle of the quadrilateral $ABCD$.

It follows from Figure 4.7 that

$$y^2 = a^2 + d^2 - 2ad\cos A = b^2 + c^2 - 2bc\cos C,$$

and so

$$(ad\cos A - bc\cos C) = \frac{1}{2}(a^2 + d^2 - b^2 - c^2). \tag{4.3.11}$$

The area $\Box$ of the quadrilateral $ABCD$ is

$$(ad\sin A + bc\sin C) = 2\,\Box. \tag{4.3.12}$$

Squaring and adding the corresponding sides of (4.3.11) and (4.3.12) yields

$$a^2d^2 + b^2c^2 - 2abcd\cos(A+C) = 4\Box^2 + \frac{1}{4}(a^2 + d^2 - b^2 - c^2)^2.$$

Consequently,

$$\begin{aligned} 16\,\Box^2 &= 4(ad+bc)^2 - (a^2 + d^2 - b^2 - c^2)^2 - 16abcd\cos^2\left\{\frac{1}{2}(A+C)\right\}. \\ &= \left[(a+d)^2 - (b+c)^2\right]\,\left[(b+c)^2 - (a-d)^2\right] \\ &\qquad -16abcd\cos^2\left\{\frac{1}{2}(A+C)\right\}. \end{aligned} \tag{4.3.13}$$

Therefore,

$$\Box^2 = (s-a)(s-b)(s-c)(s-d) - abcd\,\cos^2\left\{\frac{1}{2}(A+C)\right\}. \tag{4.3.14}$$

This is the desired result (4.3.10).

In the case of a cyclic quadrilateral $ABCD$, $A+C=\pi$ and then formula (4.3.10) reduces to the celebrated Brahmagupta formula (4.3.9).

If x and y are the diagonals of a quadrilateral, then there is another formula known as *Bretschneider's formula* for the area given by

$$(4\Box)^2 = (2xy)^2 - (a^2 + c^2 - b^2 - d^2)^2. \tag{4.3.15}$$

This expresses the square of the area in terms of the sides and diagonals. The reader is referred to Problem E1376 in the *American Mathematical Monthly*, **67** (1960) p. 291.

If a circle is inscribed in the quadrilateral, then $a+c=b+d$ and hence, the formula (4.3.15) reduces to

$$4\,\Box^2 = x^2y^2 - (ac-bd)^2. \tag{4.3.16}$$

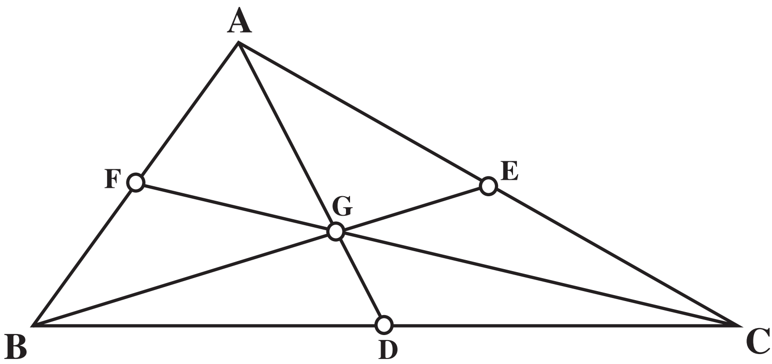


Fig. 4.9 Medians and the Centroid.

4.4 Centroid, Orthocenter and Circumcenter

The *median* is a straight line joining the vertex of a triangle ABC to the midpoint of the opposite side as shown in Figure 4.9.

The *centroid*, G is the point of intersection of three medians of the triangle ABC so that the centroid intersects each median in a ratio 2:1, that is, $AG = 2GD$.

The *orthocenter*, H is the point of intersection of three altitudes of a triangle, as shown in Figure 4.10.

Using Figure 4.10, the area of the triangle ABC is given by

$$\triangle = \frac{1}{2}\, a \cdot AD = \frac{1}{2}\, ac \sin B. \tag{4.4.1}$$

Similarly,

$$\triangle = \frac{1}{2}\, ab \sin C = \frac{1}{2}\, bc \sin A. \tag{4.4.2}$$

Using the cosine formula for a triangle ABC, we have

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ \sin^2 A = 1 - \cos^2 A &= \frac{4b^2c^2 - \left(b^2 + c^2 - a^2\right)}{4b^2c^2} \\ &= \frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{(2bc)^2} \\ &= \frac{2s(2s-2a)(2s-2b)(2s-2c)}{(2bc)^2} \\ \sin A &= \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}. \end{aligned}$$

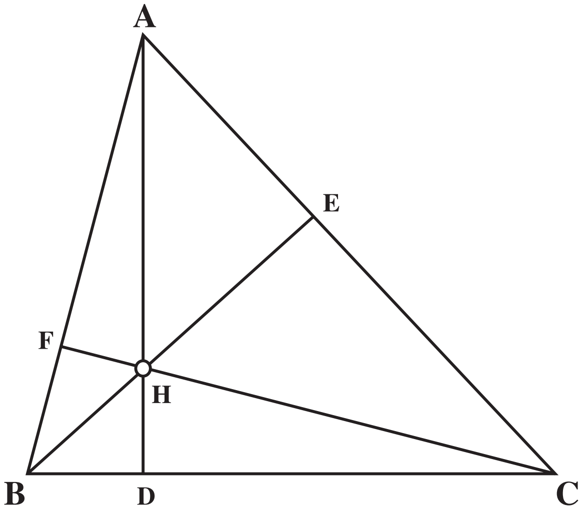


Fig. 4.10 Orthocenter of a triangle.

Thus, the area of the triangle ABC is

$$\triangle = \frac{1}{2}\,bc\sin A = \sqrt{s(s-a)(s-b)(s-c)}.$$

This is the celebrated Heron's formula (4.3.7).

Using

$$r = a\sin\frac{1}{2}B\,\sin\,\frac{1}{2}C\,\sec\frac{1}{2}A,$$

and

$$r = 4R\,\sin\frac{A}{2}\,\sin\frac{B}{2}\,\sin\frac{C}{2}.$$

The *circumcenter*, O is the center of the circumscribed circle, (or circumcircle) of a triangle. It is the intersection of the perpendicular bisectors of the three sides of a triangle, as shown in Figure 4.11.

Since the angle at the center circumcenter, $\angle BOC = 2A$, the two right angled triangles $\triangle OBP$ and $\triangle OCP$ are congruent with $\angle BOP = \angle COP = A$. Consequently,

$$R\sin A = \frac{a}{2} \quad \text{or} \quad R = \frac{1}{a}\,\text{cosec}A$$

or

$$2R = \frac{a}{\sin A}.$$

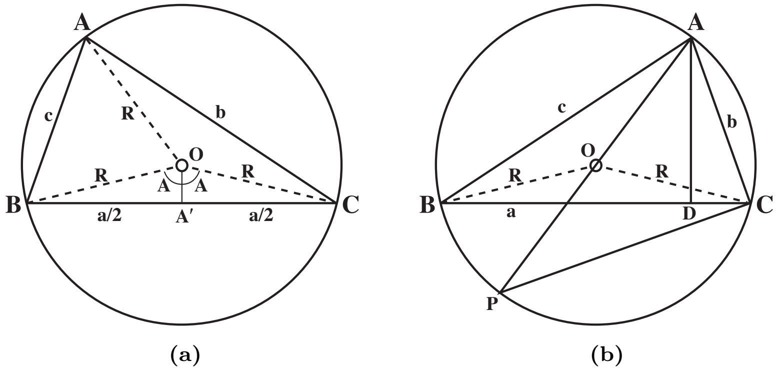


Fig. 4.11 (a) The circumcenter O and the circumcircle, (b) The circumcenter and the diameter of the circumcircle.

Similarly, we obtain

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{4.4.3}$$

Also,

$$OA' = R\cos A. \tag{4.4.4}$$

We draw a perpendicular from the vertex to the opposite BC of the triangle ABC, as shown in Figure 4.11 (b). Since two right-angled triangles ABD and APC are similar,

$$\frac{AD}{AB} = \frac{AC}{AP}.$$

Or,

$$AD = \frac{AC}{AP}\cdot AB = \frac{bc}{2R}. \tag{4.4.5}$$

Thus,

$$\triangle ABC = \frac{1}{2}BC\cdot AD = \frac{1}{2}\left(\frac{abc}{2R}\right).$$

Or,

$$4R\,\triangle ABC = abc \geq 8xyz, \quad \text{by the Lehmus inequality.} \tag{4.4.6}$$

We next draw the incircle with incenter I and inradius r, and three excircles with excenters I_a, I_b, I_c and exradius r_a, r_b, r_c associated with a triangle ABC, as shown in Figure 4.12. Then it follows that

$$\begin{aligned}\triangle ABC &= \triangle ABI_a + \triangle ACI_a - \triangle BCI_a,\\ &= \frac{1}{2}b\cdot r_a + \frac{1}{2}c\cdot r_a - \frac{1}{2}a\cdot r_a,\\ &= \frac{1}{2}(b+c-a)r_a = (s-a)r_a.\end{aligned} \tag{4.4.7}$$

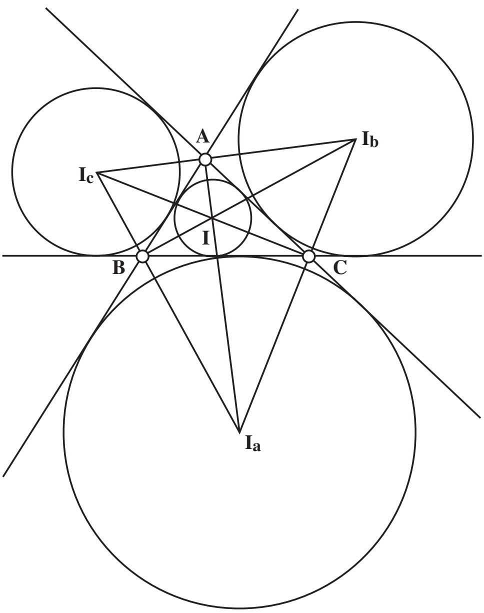


Fig. 4.12 The incircle with incenter I and excircles with excenters I_a, I_b and I_c.

Similarly,

$$\triangle ABC = (s-b)r_b \quad \text{and} \quad \triangle ABC = (r-c)r_c. \tag{4.4.8}$$

Combining (4.3.1) with (4.4.7)–(4.4.8), it turns out that

$$\triangle = rs = (s-a)r_a = (s-b)r_b = (s-c)r_c. \tag{4.4.9}$$

Using (4.3.1) and formula (4.3.6) gives

$$r^2 = \left(\frac{\triangle}{s}\right)^2 = \frac{1}{s}(s-a)(s-b)(s-c), \tag{4.4.10}$$

$$r_a^2 = \left(\frac{\triangle}{s-a}\right)^2 = \frac{s(s-b)(s-c)}{s-a}. \tag{4.4.11}$$

Similarly,

$$r_b^2 = \left(\frac{\triangle}{s-b}\right)^2 \quad \text{and} \quad r_c^2 = \left(\frac{\triangle}{s-c}\right)^2. \tag{4.4.12}$$

A simple algebra leads to the result

$$\triangle^2 = (r\, r_a r_b r_c). \tag{4.4.13}$$

Also

$$a = BC = BP + PC = r_a \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right). \tag{4.4.14}$$

Or, equivalently,

$$r_a = a \cos \frac{B}{2} \cos \frac{C}{2} \sec \frac{A}{2}, \tag{4.4.15}$$

which is, by (4.4.3)

$$r_a = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}. \tag{4.4.16}$$

Similarly,

$$r_b = 4R \sin \frac{B}{2} \cos \frac{A}{2} \cos \frac{C}{2}, \quad r_c = 4R \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}. \tag{4.4.17}$$

It follows from (4.4.6) that

$$\begin{aligned} 4R\triangle &= abc \\ &= s(s-b)(s-c) + s(s-c)(s-a) + s(s-a)(s-b) \\ &\qquad\qquad\qquad\qquad\qquad -(s-a)(s-b)(s-c) \\ &= \frac{\triangle^2}{s-a} + \frac{\triangle^2}{s-b} + \frac{\triangle^2}{s-c} - \frac{\triangle^2}{s}. \end{aligned} \tag{4.4.18}$$

which is, by (4.4.8)–(4.4.9),

$$4R\,\triangle = \triangle(r_a + r_b + r_c - r). \tag{4.4.19}$$

Canceling the common factor $\triangle$, we obtain the formula connecting the circumradius, inradius and three exradii

$$4R = (r_a + r_b + r_c) - r. \tag{4.4.20}$$

On the other hand, the sum of the reciprocals of the exradii is equal to the reciprocal of the inradius, that is,

$$\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} \right) = \frac{1}{\triangle}(3s - a - b - c) = \frac{s}{\triangle} = \frac{1}{r}. \tag{4.4.21}$$

We denote h_a, h_b and h_c are the altitudes from the vertices A, B, C on the opposite sides BC, CA, AB respectively. Hence,

$$\begin{aligned} ah_a &= 2\triangle = 2r_b(s-b) = r_b(a+c-b), \\ ah_a &= 2\triangle = 2r_c(s-c) = r_c(a+b-c). \end{aligned}$$

Or,

$$a(r_b - h_a) = r_b(b-c), \quad a(h_a - r_c) = r_c(b-c),$$

which gives, after dividing the first result by the second, and solving for h_a

$$h_a = \frac{2r_b r_c}{(r_b + r_c)}. \tag{4.4.22}$$

Similarly, we obtain two similar results for h_b and h_c:

$$h_b = \frac{2r_a r_c}{(r_a + r_c)}, \qquad h_c = \frac{2r_a r_b}{(r_a + r_b)}. \tag{4.4.23}$$

Thus, we have

$$a\,h_a = b\,h_b = c\,h_c = 2\triangle = 2rs = r(a+b+c),$$

and then

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a+b+c}{2\triangle} = \frac{s}{\triangle} = \frac{1}{r}. \tag{4.4.24}$$

This means that the sum of the reciprocals of the altitudes of a triangle is equal to the reciprocal of the inradius.

Thus, it follows from (4.4.21) and (4.4.24) that the sum of the reciprocals of the altitudes of a triangle is equal to the sum of the reciprocals of the exradii of excircles:

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}. \tag{4.4.25}$$

Finally, the value of the altitudes can be obtained as functions of exradii as follows:

$$a\,h_a = 2\triangle = 2r_s = r(a+c+b),$$
$$a\,h_a = 2\triangle = 2r_a(s-a) = r_a(b+c-a).$$

Consequently,

$$a(h_a - r) = r(b+c), \qquad a(h_a + r_a) = r_a(b+c).$$

Dividing the first result by the second and hence, solving for h_a gives

$$h_a = \frac{2rr_a}{r_a - r}. \tag{4.4.26}$$

Similarly, we can derive similar results

$$h_b = \frac{2rr_b}{r_b - r}, \qquad h_c = \frac{2rr_c}{r_c - r}. \tag{4.4.27}$$

These results (4.4.26)–(4.4.27) show that the altitude to one side of a triangle is equal to the ratio of the twice the product of the inradius and the opposite exradius to the difference of these radii.

We close this section by adding a problem of inscribing or circumscribing a regular polygon of n sides in or about a circle. We take O as the center

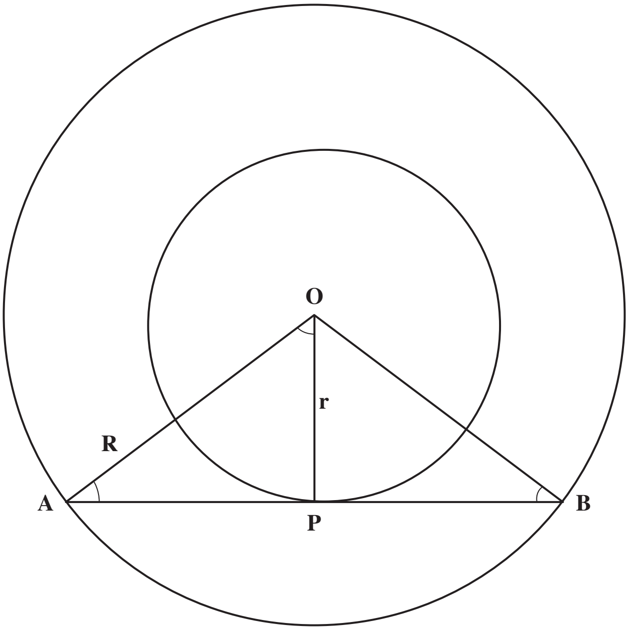


Fig. 4.13 The circumcircle and the incircle with a regular polygon of side AB.

of the circumcircle of radius R and incircle of radius r and a as the length of a side of the polygon, as shown in Figure 4.13.

We take $AB = a$ as a side of the polygon and P its point of contact with the incircle so that the angle $\angle AOB = \frac{2\pi}{n}$ and $\angle AOD = \frac{\pi}{n}$. Thus,

$$a = 2R\sin\frac{\pi}{n} = 2r\tan\frac{\pi}{n}. \tag{4.4.28}$$

The area of the triangle OAB is

$$\triangle OAB = \frac{1}{2}R^2\sin\frac{2\pi}{n} = \frac{1}{2}ar = r^2\tan\frac{\pi}{n}. \tag{4.4.29}$$

Thus, the area of the polygon is

$$\frac{1}{2}nR^2\sin\frac{2\pi}{n} = nr^2\tan\frac{\pi}{n}. \tag{4.4.30}$$

4.5 The Euler Line and the Euler Nine-Point Circle

Two plane are called *homothetic* if they are similar and similarly placed, that is, they are related by a dilation (or *homothetic transformation*) which can be defined as follows.

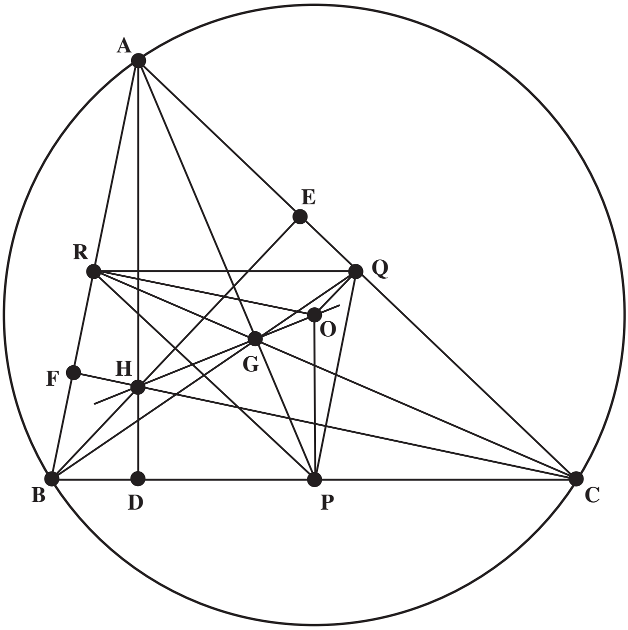


Fig. 4.14 The Euler Line HGO.

Definition. If O is a fixed point and k is a given nonzero real number, then the transformation $T(O, k)$ is called a *dilation* (or *homothetic transformation*) which transforms a point P on OP into a point P' such that $OP' = k\,OP$, where k is positive or negative provided P and P' are on the same side of O or on opposite sides of O.

The orthocenter H, centroid G and the circumcenter O of a triangle ABC lie on the same line and $HG = 2GO$. The line HGO is called the *Euler line*, as shown in Figure 4.14. Euler discovered this line in 1765.

According to the definition of the centroid, P, Q, R, are midpoints of the sides BC, CA and AB respectively of a triangle ABC, as shown in Figure 4.14. Since the ratios

$$\frac{GA}{GP} = \frac{GB}{GQ} = \frac{GC}{GR} = \frac{2}{1}. \tag{4.5.1}$$

The triangles ABC and PQR are homothetic under the homothetic transformation $T(G, -2)$. Therefore, the orthocenter O of the triangle PQR maps into the orthocenter H of the triangle ABC. Thus, it follows that H, G, O are collinear and $HG = 2GO$.

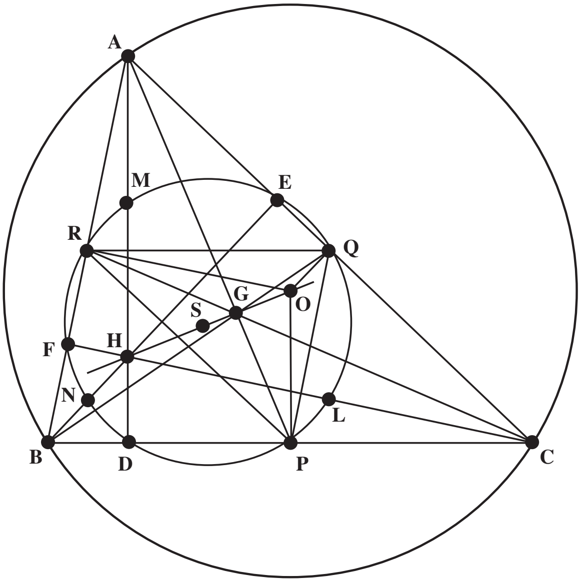


Fig. 4.15 The Euler nine-point circle.

The Euler Nine-Point Circle Theorem 4.5.1. The midpoints of the three sides of a triangle, the midpoints of the lines joining the orthocenter to the three vertices, and the feet of the three altitudes lie on a circle with the center at the midpoint of the line joining the circumcenter and the orthocenter and radius is half of the circumradius.

This nine-point circle is called the *Euler circle*. It was Poncelet who named this circle the *nine-point* circle, and this is the name commonly used in the English-speaking countries. However, the European mathematicians referred it as *Feuerback's circle*.

Proof. Making references to Figure 4.15, P, Q, R are the midpoints of the sides ABC, and L, M, N are the midpoints of HC, AH and BH respectively. Since NL and LQ are parallel to QR and RN respectively and AH is normal to the side BC, it follows that $NLQR$ is a rectangle. Similarly, $RQPB$ is a rectangle. Thus, PM, QN, RL are three diameters of a circle. Since these diameters subtends right angles at the midpoints D, E, F respectively, so the nine points D, E, F, P, Q, R, L, M N all lie on the same circle.

Clearly, both triangles PQR and MNL are homothetic to the triangle ABC under respective homothetic transformations $T\left(G, -\frac{1}{2}\right)$ and

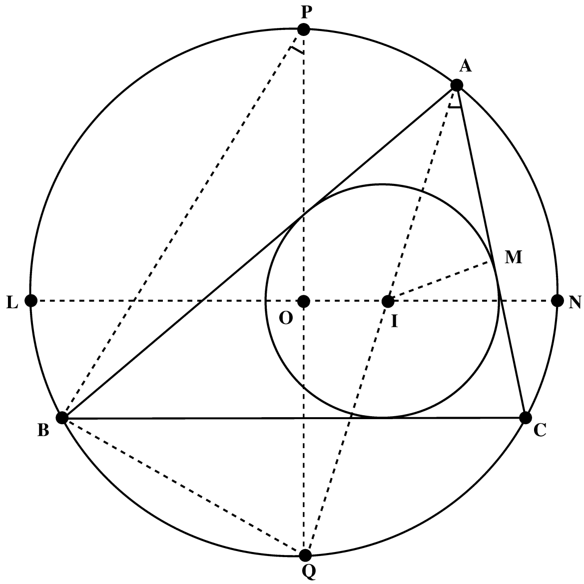


Fig. 4.16 The circumcircle and the incircle.

$T\left(H, +\frac{1}{2}\right)$. Since the ratios of homothetic transformations are $\pm\frac{1}{2}$, the radius of the nine-point circle is half of the circumradius and H, G divides the segment OS externally and internally in the ratio $2:1$, where S is the center of the nine-point circle.

Euler's Theorem 4.5.2. The square of the distance, d between the circumcenter, O, and the incenter, I of a triangle ABC is equal to the square of the circum radius, R minus the twice of the product of R and the inradius r, that is,

$$d^2 = R^2 - 2Rr. \tag{4.5.2}$$

This is called the *Euler distance formula.* Obviously, $R > 2r$. This is called the *Euler triangle inequality* which was published by Euler in 1767.

Proof. We draw a circumcenter and an incenter of a triangle ABC, as shown in Figure 4.16. The point O is the circumcenter and I is the incenter, and $OI = d$, $OP = OQ = R$, is the circumradius and $IM = r$ is the inradius.

Since $\angle IAM = \angle BPQ$, the two right-angled triangles AMI and PBQ are similar, and therefore,

$$\frac{IM}{BQ} = \frac{AI}{PQ}.$$

Or,

$$2Rr = AI \cdot BQ.$$

Since $BQ = IQ$ and so

$$2Rr = AI \cdot IQ.$$

The chord AQ of the circumscribed circle intersects the diameter through O and I, and hence,

$$AI \cdot IQ = LI \cdot IN = (R + d) \cdot (R - d). \tag{4.5.3}$$

Or

$$2Rr = R^2 - d^2. \tag{4.5.4}$$

This is the desired result.

Or, equivalently,

$$\frac{1}{r} = \frac{1}{R + d} + \frac{1}{R - d}. \tag{4.5.5}$$

This formula connects the radii r and R of the incircle and the circumcircle of a triangle and the distance d between their centers.

An argument similar to the above leads to the formulas for the distances between the circumcenter, O and excenters I_a, I_b, and I_c as

$$d_a^2 = r^2 + 2Rr_a, \quad d_b^2 = R^2 + 2Rr_b, \quad d_c^2 = R^2 + 2Rr_c. \tag{4.5.6}$$

Several distance formulas between special points can easily be obtained. We consider distances between the circumcenter O, the orthocenter H, the incenter I, the one of the excenters I_a, the center S of the Euler nine-point circle of radius $R/2$, and the centroid G, as shown in Figure 4.17. According to the Euler theorem 4.5.1, the points H, G, O lie on the Euler line, and $HG = 2GO$ and S is the center of the Euler circle so that it is the midpoint of HO. Also, $\angle IAP = \angle IAH$, $OA = R$ and $AH = 2R \cos A$,

$$AI = R\,\text{cosec}\frac{A}{2} = 4R \sin\frac{B}{2} \cos\frac{C}{2}, \quad \text{and} \quad AI_a = 4R \cos\frac{B}{2} \cos\frac{C}{2}.$$

The following results similar to those of $d = OI$, $d_a = OI_a$, $d_b = OI_b$ and $d_c = OI_c$ given by (4.5.2) and (4.5.6) can be proved:

$$d_1^2 = OH^2 = R^2(1 - 8 \cos A \cos B \cos C), \tag{4.5.7}$$

$$d_2^2 = IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C, \tag{4.5.8}$$

$$d_3^2 = IS^2 = \left(\frac{1}{2}R - r\right)^2 \quad \text{and} \quad d_4^2 = I_aS^2 = \left(\frac{1}{2}R + r_a\right)^2. \tag{4.5.9}$$

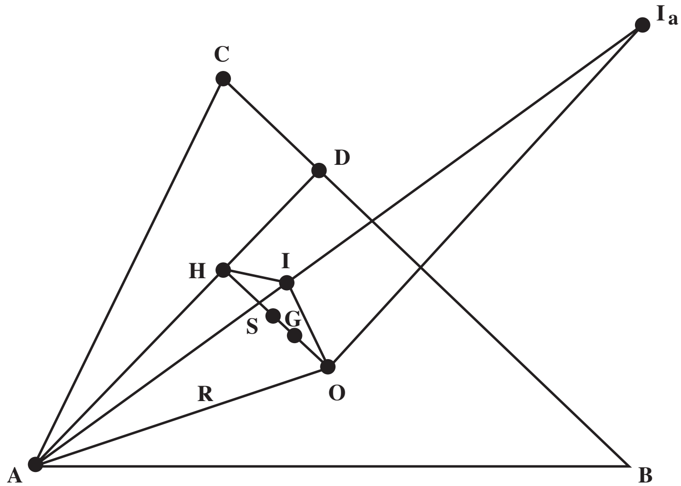


Fig. 4.17 Several special centers.

Results in (4.5.9) follow from Figure 4.16 and the results (4.5.7) and (4.5.8) as

$$\begin{aligned} IS^2 &= \frac{1}{2}IH^2 + \frac{1}{2}OI^2 - \frac{1}{4}OH^2, \\ &= r^2 + \frac{1}{2}\left(R^2 - 2Rr\right) - \frac{1}{4}R^2 = \left(\frac{1}{2}R - r\right)^2. \end{aligned}$$

Similarly,

$$I_aS = \frac{1}{2}R + r_a.$$

The last two distance formulas also clearly indicate that the incircle and the three excircles touch the Euler nine-point circle.

4.6 Euler's Work on Analytic Geometry

In the second volume of his *Introductio*, Euler made an indepth study of both algebraic and transcendental algebraic curves and proved a number of general theorems about the algebraic curves. He first introduced the general second degree equation in two dimensions with constant coefficients in the form

$$ax^2 + 2hxy + by^2 + 2fx + 2gy + e = 0, \tag{4.6.1}$$

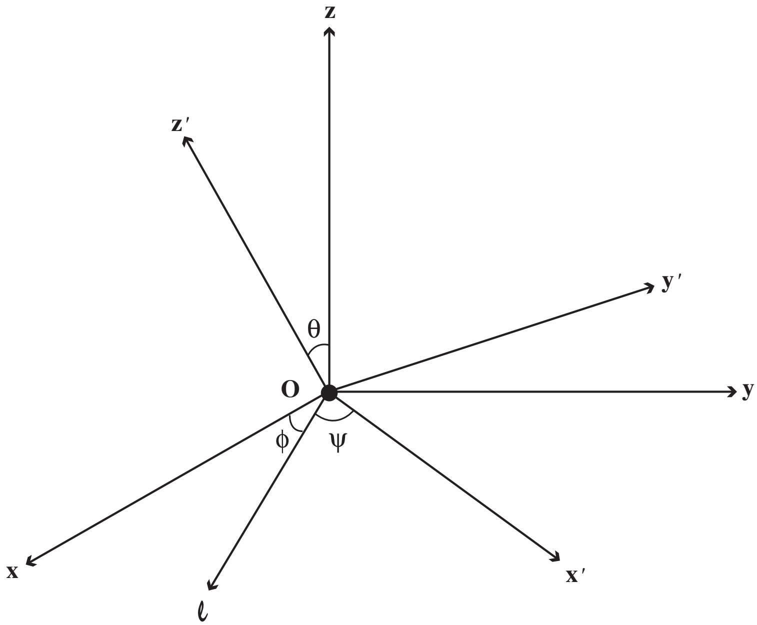


Fig. 4.18 The Euler angles ϕ, ψ and θ.

and showed that it represents various conic sections. The general equation (4.6.1) represents an ellipse, parabola or hyperbola if $ab - h^2 > 0, = 0$ or < 0 respectively, and a circle if $a = b$ and $h = 0$. He also discussed the classification of cubic and quartic algebraic curves, and introduced the parametric equations of both plane curves and surfaces, where x and y are expressed in terms of third variable t, and x, y, z are expressed in terms of two other variables u and v. For example, the parametric equations of the circle $x^2 + y^2 = a^2$ are $x = a\cos t$ and $y = a\sin t$, and $x = at^2$, $y = 2at$ are the parametric equations of the parabolic $y^2 = 4ax$. Similarly, the parametric equations of the ellipse $b^2x^2 + y^2a^2 = a^2b^2$ are $x = a\cos t$ and $y = b\sin t$. For hyperbola $x^2/a^2 - y^2/b^2 = 1$, the parametric equations are $x = a\cosh t$ and $y = b\sinh t$.

Euler also investigated three dimensional coordinate geometry by introducing the general second degree equation in three variables with constant coefficients in the form

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + kz + m = 0. \qquad (4.6.2)$$

Euler first introduced the transformation from the $0xyz$-system to the $0x'y'z'$ system whose equations are represented (see Figure 4.18) in terms of the *Euler angles* ϕ, ψ and θ. These angles are considered as the angles through which the former must be successively rotated about the axes of the latter so that in the end the two systems coincide. The angle ϕ is measured in the xy-plane from the x-axis to the line of nodes ℓ which is the line of intersection of the planes $0xy$ and $0x'y'$. The Cartesian coordinates x, y, z and x', y', z' are related by the relations

$$\begin{aligned} x = x' &(\cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi) \\ &-y'(\cos\psi\sin\phi + \cos\theta\sin\psi\sin\phi) + z'\sin\theta\sin\phi, \end{aligned} \quad (4.6.3)$$

$$\begin{aligned} y = x' &(\sin\psi\cos\phi + \cos\theta\cos\psi\sin\phi) \\ &-y'(\sin\psi\sin\phi - \cos\theta\cos\psi\sin\phi) - z'\sin\theta\sin\phi, \end{aligned} \quad (4.6.4)$$

$$z = x'\sin\theta\sin\phi + y'\sin\theta\cos\phi + z'\cos\theta. \quad (4.6.5)$$

Euler used this transformation to transform (4.6.2) to cannonical forms and obtained *seven* different cases: cone, cylinder, ellipsoid, hyperboloid of one and two sheets, parabolic cylinder and hyperbolic paraboloid - the last of these was his own discovery. He also discovered that the degree of a curve is invariant under a linear transformation.

Isaac Newton proved that the general third degree algebraic equation representing cubic curves in the form

$$ax^3 + bx^3y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + jy + k = 0, \quad (4.6.6)$$

can be transformed by a change of axis, into one of the following four forms such as (i) $xy^2 + ey = f(x)$, (ii) $xy = f(x)$, (iii) $y^2 = f(x)$, and (iv) $y = f(x)$, where $f(x) = ax^3 + bx^2 + cx + d$. Newton's work on third degree plane curves stimulated much other work on higher degree plane curves. The classification of third and fourth degree curves also continued to interest mathematicians and physicists of the eighteenth and nineteenth centuries. It became evident from the study of cubic curves and the curves of higher-degree that equations of these curves exhibit many special features such as singular points, inflection points, and multiple points. In general, $f(x, y) = 0$ represents a general conic section, where f is a polynomial in x and y. In particular, if $f(x, y)$ has no first degree terms and contains only second degree terms such as $f(x, y) = ax^2 + 2hxy + by^2$, then $f(x, y) = 0$ represents several conic sections depending on nature of a, h and b including two different straight lines.

In his letter of 1643, Fermat provided a brief sketch of his major ideas on analytic geometry of three dimensions and then introduced the plane curves in general and curves on surfaces including cylindrical surfaces, elliptic paraboloid, hyperboloid of two sheets and ellipsoids.

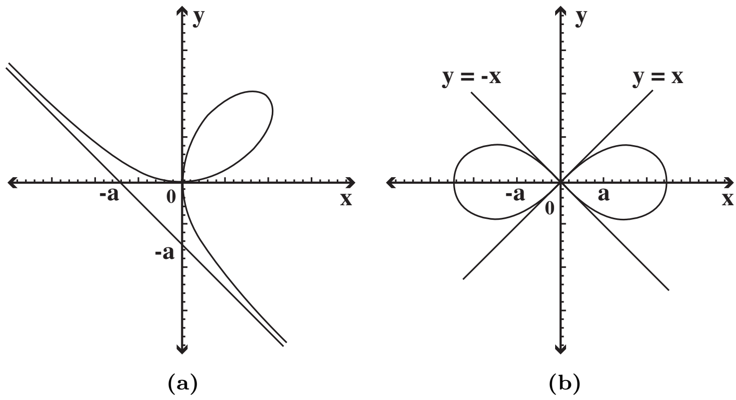


Fig. 4.19 (a) Folium of Descartes, (b) The Bernoulli lemniscate.

For example, the plane curve

$$f(x,y) \equiv x^3 + y^3 - 3axy = 0, \tag{4.6.7}$$

represents *Cartesian leaf* or *folium of Descartes* (see Figure 4.19 (a)). It has tangent at the origin which is its only singularity. The asymptote is a straight line $x + y + a = 0$.

The plane curve

$$f(x,y) \equiv (x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0, \qquad a > 0, \tag{4.6.8}$$

represents the *Bernoulli lemniscate* (see Figure 4.19 (b)). Its second degree term $x^2 - y^2 = 0$ or $y = \pm x$ represent the tangents at the origin. In polar coordinates r, θ, the equation of the Bernoulli lemniscate is $r^2 = a^2 \cos 2\theta$.

The plane curve

$$f(x,y) \equiv ay^2 - x^3 = 0, \tag{4.6.9}$$

represents the *semicubical parabola* with a cusp at the origin as shown in Figure 4.20 (a) and $y^2 = 0$ is the equation of the two coincident tangents to the curve.

The equation

$$f(x,y) \equiv \left(x^2 - y^2 - ax\right)^2 - a^2\left(x^2 + y^2\right) = 0, \tag{4.6.10}$$

represents the *Cardioid* as shown in Figure 4.20 (b). Its equation in polar coordinates is $r = a(1 + \cos\theta)$, $a > 0$.

The equation

$$f(x,y) \equiv a\left(y^3 - 3x^2 y\right) - x^4 = 0, \tag{4.6.11}$$

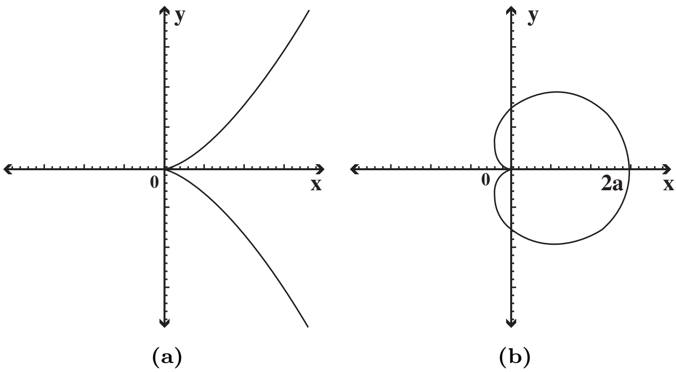


Fig. 4.20 (a) The semicubical parabola, (b) The Cardioid.

represents a *plane curve* as shown in Figure 4.21 (a) with a triple point at the origin and with the three tangents with equations $a\left(y^3 - 3x^2y\right) = 0$ or $y = 0$ and $y = \pm\sqrt{3}x$.

The equation

$$f(x, y) \equiv (x + a)x^2 + (x - a)^2y^2 = 0, \tag{4.6.12}$$

represents the *Strophoid* as shown in Figure 4.21 (b) with asymptote $x = a$.

The equation

$$f(x, y) \equiv a\left(y^4 - x^2y^2\right) - x^5 = 0, \tag{4.6.13}$$

represents a *plane curve* as shown in Figure 4.22 (a) with a quadruple point at $(0, 0)$ which is a combination of a cusp and a node. It has four tangents with equations $y = 0$, 0, $y = \pm x$.

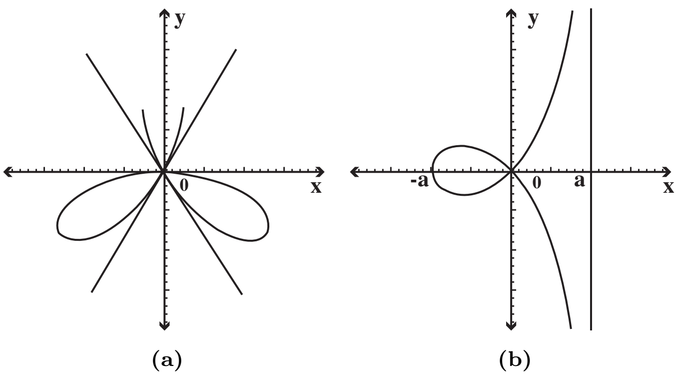


Fig. 4.21 (a) A plane curve with a triple point at $(0, 0)$, (b) The Strophoid.

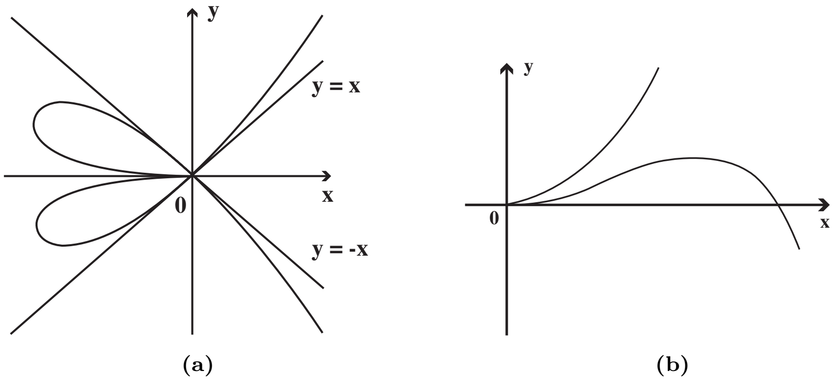


Fig. 4.22 (a) A plane curve with a quadruple point at $(0, 0)$, (b) A plane curve with a cusp at the origin.

The equation

$$f(x, y) \equiv (y - x^2)^2 - x^5 = 0, \tag{4.6.14}$$

represents a *plane curve* as shown in Figure 4.22 (b) with a cusp at the origin. Both branches of the curve lie on the same side of the double tangent $(y = 0)$.

In 1739, Euler gave many examples of such curves with a cusp which is also called a *stationary point* because a point moving along the curve should come to stop before continuing its motion at a cusp.

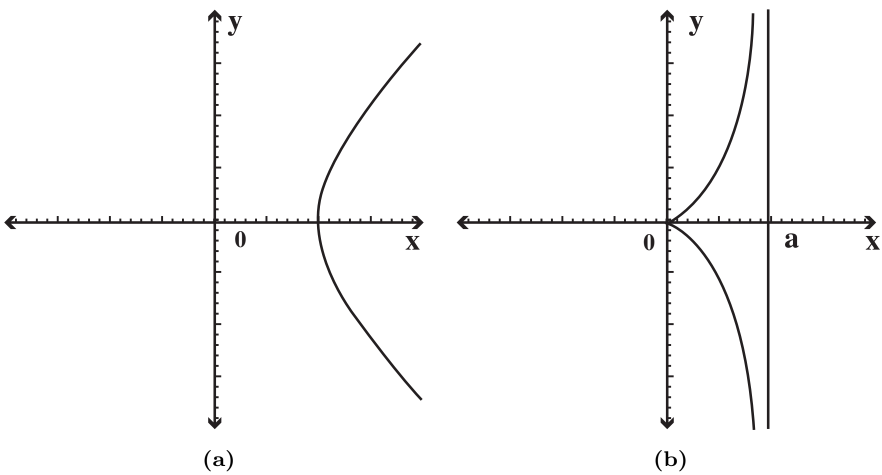


Fig. 4.23 (a) A plane curve with imaginary tangents, (b) The Cissoid of Diocles.

The equation

$$f(x, y) \equiv y^2 - x^2(2x - 1) = 0, \tag{4.6.15}$$

represents a curve as shown in Figure 4.23 (a) which has a double *imaginary tangent* with equation $y^2 = -x^2$. It has a *conjugate point* at the origin. When the two tangents are imaginary, the double point is referred to as *conjugate point.*

The equation

$$x^3 + (x - a)y^2 = 0, \tag{4.6.16}$$

with an asymptote $x = a$ represents the *Cissoid of Diocles* as shown in Figure 4.23 (b).

Finally, the geometrical locus of all centers of curvature of a given curve C is called the *evolute* E of C. For a given curve C with equation $y = f(x)$, the equation of its evolute in the parametric form is given by

$$x = t - \frac{f'(t)\left[1 + f'^2(t)\right]}{f''(t)}, \quad y = f(t) + \frac{1 + f'^2(t)}{f''(t)}, \tag{4.6.17}$$

provided $f''(t) \neq 0$.

For example, the parametric equations of the evolute of a parabola $y = ax^2$ are given by

$$x = -4a^2t^3, \qquad y = \frac{1}{2a} + 3at^2. \tag{4.6.18}$$

Eliminating t, we obtain the equation of the evolute of the given parabola

$$y = \frac{1}{2a} + 3a\left(\frac{x}{4a^2}\right)^{3/2}. \tag{4.6.19}$$

Both the parabola and its evolute are shown in Figure 4.24 (a).

Similarly, the evolute of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{4.6.20}$$

is the *astroid* as shown in Figure 4.24 (b).

4.7 Euler's Work on Differential Geometry

Differential geometry is concerned with the study of curves and surfaces using differential calculus and analysis. Many new curves and surfaces and their equations were introduced and their properties were studied in detail.

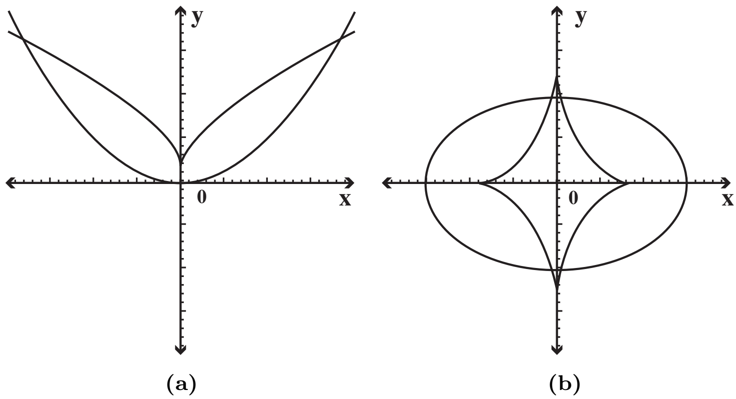


Fig. 4.24 (a) Parabola and its Evolute, (b) An Evolute of an ellipse.

It was Alexis-Claude Clairaut studied the theory of space curves which represents the first major development of three-dimensional differential geometry. In his famous book on *Recherche sur les courbes à double courbure* (*Research on the curves of double curvature*) published in 1731, Clairaut not only gave the equations of some surfaces, but pointed out that a space curve is the intersection of two surfaces. Furthermore, he introduced the fundamental idea that a space curve has two curvatures. Although some of the quadratic surfaces including sphere, cylinder, paraboloid, ellipsoid and hyperboloid of two sheets were known before 1700, Clairaut in his book of 1731, presented equations of some of these quadric surfaces. More precisely, he proved that an equation that is homogeneous in x, y, and z (all terms of the equation one of the same degree) represents a cone with vertex at the origin of the rectangular Cartesian coordinate system. In 1732, Jacob Hermann (1678-1733) discovered that the general equation of the form $x^2 + y^2 = f(z)$ represents a surface of revolution about the z-axis. He also gave the transformation from rectangular to polar coordinates.

Although Clairaut and Hermann first developed the theory of space curves and surfaces, Euler provided the next major step in the differential geometry of space curves and surfaces during 1748-1760. In the second part of his *Introductio* of 1748, Euler studied coordinate geometry and differential geometry of planar curves completely and spatial curves briefly. He also expanded the use of polar coordinates and used trigonometric notations explicitly. Motivated by his fundamental ideas and use of curves and surfaces in analytical mechanics, Euler made some major contributions to differential geometry. He did a great deal of original work on particle dynamics and then on the dynamics of rigid bodies in his first famous treatise *Mechanica* which was published in 1736. In this book he used the currently adopted polar coordinates to formulate the radial and normal components of acceleration of a particle moving along a plane curve in the form

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2, \qquad a_\theta = r\,\frac{d^2\theta}{dt^2} + 2\,\frac{dr}{dt}\cdot\frac{d\theta}{dt}, \tag{4.7.1ab}$$

where the polar coordinates r, θ are the functions of time t.

Motivated by his fundamental work on the skew elastica, that is, the form assumed by an initially straight band, when under pressure at the ends, it is bent and twisted of a skew curve, Euler developed a complete theory of skew curves in 1774-1775.

Although Huygens geometrically derived the formula for the radius of curvature $\rho(x)$ of a plane curve, $y = f(x)$ in the form

$$\frac{1}{\rho(x)} = \kappa(x) = \frac{y''}{(1+y'^2)^{3/2}}, \tag{4.7.2}$$

Euler gave an analytical proof of this result in 1764. Indeed, Euler represented space curves by the parametric equations $x = x(s)$, $y = y(s)$ and $z = z(s)$, where s is the curve length. From this parametric representation, he obtained

$$dx = p\,ds, \quad dy = q\,ds, \quad dz = r\,ds, \tag{4.7.3}$$

where p, q, r are the direction cosines at each point with $p^2 + q^2 + r^2 = 1$. To study the properties of the space curves, Euler introduced the definition of the radius of curvature of a curve by

$$\rho = \frac{ds'}{ds}, \tag{4.7.4}$$

where ds' is the arc or the angle between the two neighboring tangents of the points that are ds apart along the curve. He then derived an analytical formula for the radius of curvature

$$\rho = \frac{ds^2}{\sqrt{(d^2x)^2 + (d^2y)^2 + (d^2z)^2}} = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}}. \tag{4.7.5}$$

The plane passing through the arc ds' and the origin is called the *Euler osculating plane* at (x, y, z) whose equation is introduced by Euler in the form

$$x\,(r\,dq - q\,dr) + y\,(p\,dr - r\,dp) + z\,(q\,dp - p\,dq) = t, \tag{4.7.6}$$

where t is determined by the point (x, y, z) on the curve through which the osculating plane passes. In modern vector notation, equation (4.7.6) can be written as

$$(\mathbf{R} - \mathbf{r}) \cdot (\mathbf{r}' \times \mathbf{r}'') = 0, \tag{4.7.7}$$

where $\mathbf{r} = \mathbf{r}(s)$ is the position vector of some point in space of the point on the curve at which the osculating plane is determined, and $\mathbf{R}(s)$ is the position vector of any point of the osculating plane.

Clairaut recognized that *curvature* and *torsion* of a space curve at a point (x, y, z) are two fundamental geometrical properties of the space curve. The former, introduced by Euler, is a measure of the rate at which the curve is turning away from the tangent line at (x, y, z). The torsion of the curve at (x, y, z) is also a measure of the rate at which the curve is twisting out of the osculating plane at (x, y, z). Michel-Ange Lancret (1774-1807), who was a student of famous French geometer, Gaspard Monge (1746-1818), also introduced the idea of torsion in differential geometry. In 1806, Lancret formulated three principal directions at any point of a twisted curve and they are *tangent*, *normal* and *binormal*, where the tangent vector $\mathbf{t} = \mathbf{r}' = (x', y', z')$ can be defined in the same way as for a plane curve, the normal to the curve that lies in the osculating plane is the *principal normal*, $\mathbf{n}$ and the perpendicular to the osculating plane, the *binormal*, $\mathbf{b}$ is the third principal direction. These vectors $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ at a point (x, y, z) are the unit vectors in the positive directions and they constitute what is called the *trihedral* at the point (x, y, z). They satisfy the following relations:

$$\mathbf{t}^2 = \mathbf{n}^2 = \mathbf{b}^2 = 1, \quad \mathbf{n} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{t} = \mathbf{t} \cdot \mathbf{n} = 0, \tag{4.7.8}$$

$$\mathbf{t} = \mathbf{n} \times \mathbf{b}, \qquad \mathbf{n} = \mathbf{b} \times \mathbf{t}, \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n}, \quad [\mathbf{t}\,\mathbf{n}\,\mathbf{b}] = 1. \tag{4.7.9}$$

Thus, the torsion is the rate of change of the direction of the binormal with respect to arc length, ds.

Subsequently, two French mathematicians, Joseph Alfred Serret (1818-1885) in 1851 and Jean Frénet (1816-1900) in 1852 formulated a set of three fundamental formulas, universally known as the *Serret-Frénet formulas*, for the tangent vector $\mathbf{t}$, the principle normal vector $\mathbf{n}$ and the binormal vector $\mathbf{b}$ in the form

$$\mathbf{t}' = \kappa\,\mathbf{n}, \quad \mathbf{n}' = \tau\,\mathbf{b} - \kappa\,\mathbf{t}, \quad \text{and} \quad \mathbf{b}' = -\tau\,\mathbf{n}, \tag{4.7.10}$$

where a prime denotes the derivative with respect to arc length s, κ is the *curvature*, and τ is the *torsion* of the curve. These derivatives can be expressed in terms of *Gaston Darboux* (1842-1917) *vector*

$$\mathbf{d} = \tau\mathbf{t} + \kappa\mathbf{b}. \tag{4.7.11}$$

It is easy to verify that

$$\boldsymbol{\alpha}' = \mathbf{d} \times \boldsymbol{\alpha}, \tag{4.7.12}$$

where $\boldsymbol{\alpha} = \mathbf{t}$ or $\mathbf{n}$ or $\mathbf{b}$.

The *circular helix* has the parametric equations

$$x = a\cos u, \quad y = a\sin u, \quad z = cu. \tag{4.7.13}$$

Or,

$$x^2 + y^2 = a^2, \qquad \frac{y}{x} = \tan\frac{z}{c}, \tag{4.7.14}$$

which is the curve of intersections of two surfaces: the *circular cylinder*

$$r = a \qquad \text{or} \qquad x^2 + y^2 = a^2 \tag{4.7.15}$$

and the *helicoid*

$$z = c\theta \qquad \text{or} \qquad \frac{y}{x} = \tan\frac{z}{c} \tag{4.7.16}$$

which has the shape of a staircase.

We differentiate $\mathbf{r} = (a\cos u,\, a\sin u,\, cu)$ with respect to s to obtain

$$\mathbf{t} = u'(-a\sin u,\, a\cos u,\, c). \tag{4.7.17}$$

Since this must be a unit vector, we find

$$u' = \frac{1}{\sqrt{a^2 + c^2}}. \tag{4.7.18}$$

The Serret-Frénet formulas give

$$\kappa\,\mathbf{n} = \mathbf{t}' = u'^2(-a\cos u,\, -a\cos u,\, 0) = -au'^2(\cos u,\, \sin u,\, 0). \tag{4.7.19}$$

Thus,

$$\kappa = au'^2 = \frac{a}{a^2 + c^2}. \tag{4.7.20}$$

Hence,

$$\mathbf{n} = -(\cos u,\, \sin u,\, 0), \qquad \mathbf{b} = \mathbf{t} \times \mathbf{n} = u'(c\sin u,\, c\cos u,\, a), \tag{4.7.21}$$

$$-\tau\,\mathbf{n} = \mathbf{b}' = u'^2\, c\,(\cos u\, \sin u,\, 0), \tag{4.7.22}$$

so that

$$\tau = c\,u'^2 = \frac{c}{(a^2 + c^2)}. \tag{4.7.23}$$

It follows from (4.7.20) and (4.7.23) that the curvature and torsion of a circular helix are both constant. Conversely, if a curve whose curvature and torsion are constant, it is a circular helix including the straight line

($\kappa = 0$ and $a = 0$) and the circle ($\tau = 0$ and $c = 0$) as limiting cases. In case of a twist about a certain line (*screw axis*) in the direction of the Darboux vector, $\mathbf{d} = \kappa\mathbf{b} + \tau\mathbf{t}$ through a point whose position vector is $\mathbf{r} + a\,\mathbf{n}$, where a is the radius of the circular cylinder containing the helix. Hence, a is obtained by eliminating c from (4.7.20) and (4.7.23) as

$$a = \frac{\kappa}{\kappa^2 + \tau^2}. \tag{4.7.24}$$

For a plane curve, $\tau = 0$ and $a = \frac{1}{\kappa} = \rho$, the position vector $\mathbf{r} + \rho\,\mathbf{n}$ and the Darboux vector becomes $\kappa\mathbf{b}$ which is normal to the plane of the curve.

When κ and τ are specified at each point of a curve, the shape of the curve, except for its location, is uniquely determined in three dimensional space. Conversely, a curve can be reconstructed from its curvature and torsion except for position in space.

In his famous paper 'Recherches sur la courbure des surfaces' published in 1760, Euler established his theory of surfaces which can be regarded as a landmark contribution to differential geometry as a new branch of geometry. Indeed, he may be considered as a founder of differential geometry. Defining the equation of a surface by $z = f(x, y)$, Euler introduced new standard notations

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}. \tag{4.7.25}$$

It is appropriate to quote here his own words:

"I begin by determining the radius of curvature of any plane section of a surface; then I apply this solution to sections which are perpendicular to the surface at any given point; and finally I compare the radii of curvature of these sections with respect to their mutual inclination, which puts us in a position to establish a proper idea of the curvature of surfaces."

He then discovered two principal normal sections of a surface and the principal curvatures κ_1 and κ_2. One of his results, the so called *Euler's equation*, gives the curvature κ of any other normal section making an angle α with one of the sections with the principal curvature in the form

$$\kappa = \kappa_1 \cos^2 \alpha + \kappa_2 \sin^2 \alpha. \tag{4.7.26}$$

It was Euler who first considered the subject of *developable surfaces* (for example, a cylinder or a cone), that is, surfaces that can be deformed into a plane without distortion such as stretching or tearing. A surface is called a *ruled surface* (for example, a cylinder, cone, hyperboloid or hyperbolic paraboloid), if it can be generated by the motion of a straight line in space. In 1775, Monge used an intuitive geometrical argument to demonstrate that a developable surface is a ruled surface on which two consecutive lines are parallel or concurrent and that any developable surface is equivalent to that formed by the tangents to a three dimensional space curve. However, the converse is not necessarily true, that is, a ruled surface is *not* a developable surface. Monge also gave a general representation of developable surfaces with their equations in the form

$$z = x\left[F(q) - qF'(q)\right] + f(q) - qf'(q), \tag{4.7.27}$$

where $q = \frac{\partial z}{\partial y}$ and such surfaces except for cylinders normal to $x - y$ plane, always satisfy the partial differential equation

$$z_{xx}z_{yy} - z_{xy}^2 = 0. \tag{4.7.28}$$

Monge then investigated the general form of ruled surfaces and gave a general representation for them. He also proved that developed surfaces are a particular case of ruled surfaces. In addition to his celebrated work *Géométrie Desciptive* published in 1798, Monge made major contributions to the theory of nonlinear partial differential equations. He not only gave the geometric interpretation, but introduced the new idea of *characteristic curves*. Subsequently, the theory of characteristics and integrals as envelopes became very significant subject of research in partial differential equations.

Motivated by the study of ruled and developable surfaces in differential geometry, Euler introduced the parametric equations of surfaces as

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v), \tag{4.7.29}$$

where u and v represent two real parameters and he investigated the conditions under which (4.7.29) become a developable surface on a plane. Following a new pioneering work in theoretical cartography by J. H. Lambert (1728-1877), Euler made also some important contributions to the subject and actually designed a map of whole Russia. In his paper presented to the St. Petersburg Academy in 1768, Euler used the ideas of complex functions to develop a general method of representing conformal transformations from one place to another.

For example, the equations

$$x = a\sin\phi\cos\theta, \quad y = a\sin\phi\sin\theta, \quad z = a\cos\phi, \tag{4.7.30}$$

represent parametrically the sphere with radius a and center at the origin. The parameters ϕ and θ are the colatitude and longitude of a point on the sphere.

Similarly, the parametric equations of a circular cylinder erected on the circle in the $x - y$ plane with radius a and center at the origin are

$$x = a\cos u, \quad y = a\sin v, \quad z = v. \tag{4.7.31}$$

The next great step in differential geometry was made by Friedrich Gauss and Bernhard Riemann. In his third major treatise '*Disquisitiones generales circa superficies curvas*' (*General Investigations of curved surfaces*) published in 1827, Gauss provided a remarkable new treatment of the differential geometry of surfaces in three dimensional spaces. Based on the pioneering work of Euler, Gauss used the idea that the coordinates of any point on a surface can be represented in terms of two parameters u and v represented by (4.7.29). From these parametric representations, he wrote

$$\begin{aligned} dx &= \left(\frac{\partial x}{\partial u}\right) du + \left(\frac{\partial x}{\partial v}\right) dv, \quad dy = \left(\frac{\partial y}{\partial u}\right) du + \left(\frac{\partial y}{\partial v}\right) dv, \\ dz &= \left(\frac{\partial z}{\partial u}\right) du + \left(\frac{\partial z}{\partial v}\right) dv, \end{aligned} \tag{4.7.32}$$

and he then was able to derive very simple what he himself described as "almost everything that the illustrious Euler was the first to prove about the curvature of curved surfaces". In particular, he proved that the total curvature κ is the reciprocal of the product of the two principal radii of curvatures κ_1 and κ_2 at a point (x, y, z) which were introduced by Euler.

The fundamental quantity on any surface S is the element of arc length ds given by

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{4.7.33}$$

Using (4.7.32), Gauss expressed (4.7.33) in the form

$$ds^2 = E(u, v)\, du^2 + 2F(u, v)\, dudv + G(u, v)\, dv^2, \tag{4.7.34}$$

where

$$E(u, v) \equiv \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \tag{4.7.35}$$

$$F(u, v) \equiv \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \cdot \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \cdot \frac{\partial z}{\partial v} \tag{4.7.36}$$

$$G(u, v) \equiv \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2. \tag{4.7.37}$$

The expression on the right-hand side of (4.7.34) is a quadratic form, that is, a homogeneous polynomial of degree two in du and dv. This is known as the *first fundamental differential quadratic form* of the surface S.

The *discrement*, D^2 of the quadratic form (4.7.34) is defined by

$$D^2 = EG - F^2, \tag{4.7.38}$$

where $D = \sqrt{EG - F^2}$ is positive at every regular point of the surface.

It follows from (4.7.34) that the differentials of arc of an arbitrary u-curve and an arbitrary v-curve are given respectively by

$$ds = \sqrt{E}\,du, \qquad ds = \sqrt{G}\,dv, \tag{4.7.39}$$

for, in the first case, v=constant, or $dv = 0$; and in the second case, $du = 0$.

If the angle between the directed parametric curves on S at any point P is ω, then, for $0 < \omega < \pi$,

$$\cos\omega = \frac{F}{\sqrt{EG}} \quad \text{and} \quad \sin\omega = \frac{D}{\sqrt{EG}}. \tag{4.7.40}$$

Thus, when $F = 0$ at P, these curves intersect orthogonally. This means that the parametric curves on the surface S form an orthogonal system if and only if $F = 0$.

In general, the *angle* between two curves on a surface is another fundamental quantity. A curve on the surface is determined by a relation between u and v, and so, equations (4.7.29) represent the parametric representation of a curve. In the language of differential geometry, the direction of a curve originating from the point (u, v) is given by the ratio $du : dv$. If two curves C and C' or two directions originating from (u, v), one given by $du : dv$ and the other by $du' : dv$, and if θ is the angle between these two curves, Gauss proved that

$$\cos\theta = \frac{E\,dudu' + F(dudv' + du'dv) + G\,dvdv'}{\sqrt{Edu^2 + 2Fdudv + Gdv^2}\,\sqrt{Edu'^2 + 2Fdu'dv' + Gdv'^2}}, \tag{4.7.41}$$

$$\sin\theta = \frac{D|dudv' - dvdu'|}{\sqrt{Edu^2 + 2Fdudv + Gdv^2}\,\sqrt{Edu'^2 + 2Fdu'dv' + Gdv'^2}}. \tag{4.7.42}$$

Obviously, (4.7.41) implies that a necessary and sufficient condition that the curves C and C' intersect orthogonally at the point P is that, at P,

$$E\,dudu' + F(dudv' + dvdu') + G\,dvdv' = 0. \tag{4.7.43}$$

The area of a closed region on the surface is given by the double integral

$$A = \iint Ddudv. \tag{4.7.44}$$

In other words, $dA = Ddudv$ is elementary area of the surface referred to the curvilinear coordinates (u, v).

Gauss also introduced the *second fundamental quadratic form* of a surface is given by

$$e\,du^2 + 2f\,dudv + g\,dv^2, \tag{4.7.45}$$

where e, f, and g are given by

$$e = \frac{1}{D}\begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},\ f = \frac{1}{D}\begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix},\ g = \frac{1}{D}\begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}. \tag{4.7.46}$$

The discriminant of the second fundamental form (4.7.45) is defined by

$$d^2 = e\,g - f^2, \tag{4.7.47}$$

which may be positive, negative or zero.

Gauss introduced the fundamental idea of curvature at a point P of an arbitrary curve on the surface S, that is, in general, related to the normal curvature at P in the direction of the curve. The *normal curvature* at P in the direction (dv/du) is given by

$$\kappa = \frac{e\,du^2 + 2f dudv + g\,dv^2}{E\,du^2 + 2F dudv + G\,dv^2}. \tag{4.7.48}$$

Or, equivalently, setting $\lambda = \frac{dv}{du}$,

$$\kappa = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2}, \tag{4.7.49}$$

where e, f, g, E, F, G are evaluated at the point P.

The two directions in which the normal curvature has its extrema are known as the *principal normal curvatures* at the point P. In other words, they are the directions for which the $\frac{d\kappa}{d\lambda} = 0$ so that the values of λ which define them are the solutions of the quadratic equation

$$(Fg - Gf)\lambda^2 + (Eg - Ge)\lambda^2 + (Ef - Fe) = 0, \tag{4.7.50}$$

Therefore, the principal direction $\lambda = (dv/du)$ at the point P are given by the differential equation

$$(Ef - Fe)\,du^2 + (Eg - Ge)\,dudv + (Fg - Gf)\,dv^2 = 0, \tag{4.7.51}$$

where E, F, G, e, f, g are evaluated at the point P. They are the values of κ for which the two corresponding values of λ, obtained from (4.7.49) are equal. The quadratic equation for λ is obtained from (4.7.49) in the form

$$\lambda^2(G\kappa - g) + 2(F\kappa - f)\lambda + (E\kappa - e) = 0. \tag{4.7.52}$$

The two roots of this equation are equal provided its discriminant is zero which gives the quadratic equation for κ whose values are the *principal normal curvatures* so that

$$(EG - F^2)\kappa^2 - (E\,g - 2Ff + G\,e)\kappa + (e\,g - f^2) = 0, \tag{4.7.53}$$

so that the sum and product of the two *principal normal curvatures* κ_1 and κ_2 are given by

$$\kappa_1 + \kappa_2 = \frac{1}{D^2}(E\,g - 2Ff + G\,e), \tag{4.7.54}$$

$$\kappa_1\kappa_2 = \frac{(e\,g - f^2)}{(EG - F^2)} = \frac{d^2}{D^2}. \tag{4.7.55}$$

The product $\kappa = \kappa_1\kappa_2 = (d/D)^2$ is called the *total curvature* (or *Gaussian curvature*) of the surface at the point P. The sum $(\kappa_1 + \kappa_2)$ is known as the *mean curvature* of the surface at P.

In differential geometry, surfaces are classified according to the surface of distinct tangent planes. So, a surface is called a *plane*, a *developable surface* or an *ordinary surface* according as the surface has a single tangent plane, a one-parameter family of distinct tangent planes, or a two-parameter family of distinct tangent planes. The directions at the point $P : (u, v)$ of a surface S in which the tangent plane has contact of at least the second order are the directions at P for which

$$e\,du^2 + 2f\,dudv + g\,dv^2 = 0. \tag{4.7.56}$$

These directions, if they exist, are known as the *asymptotic directions* at P. If e, f, g are not all zero at a point P and $d^2 = eg - f^2$, the equation (4.7.56) defines at P two directions which are real and distinct, real and coincident, or imaginary according as $d^2 < 0$, $d^2 = 0$, or $d^2 > 0$ at P.

In order to classify points on a surface, the following criteria are used. A point P on S is called a *planar* point if $e = 0$, $f = 0$, and $g = 0$. A nonplanar point P is called *elliptic*, *parabolic* and *hyperbolic* according as $d^2 > 0$, $d^2 = 0$, or $d^2 < 0$ at P. The asymptotic directions at P are the directions at P in which the normal curvature is zero.

The fact that $\kappa = d^2/D^2$ has the same sign as d^2 can be interpreted as follows. A nonplanar point P is an elliptic, parabolic and hyperbolic point according as $\kappa > 0$, $\kappa = 0$, or $\kappa < 0$ at the point P. If P is an elliptic point, $\kappa > 0$ and κ_1 and κ_2 are of the same sign. Then κ is always of this sign, since κ varies between κ_1 and κ_2. Thus, the centers of curvatures of the normal sections at P all lie on one half of the surface normal, and so, the surface in the neighborhood of P lies on one side of the tangent plane.

If P is a parabolic point, $\kappa = 0$ and either $\kappa_1 = 0$ or $\kappa_2 = 0$. In other words, the single asymptotic direction coincides with a principal direction. Except along this direction, the surface is on one side of the tangent plane.

If P is a hyperbolic point, $\kappa < 0$ and hence, κ_1 and κ_2 are opposite in sign. In this case, κ is positive for certain directions, negative for other directions, and zero in the two asymptotic directions. Thus, the surface lies in part on the one side, and in part on the other side of the tangent plane, and cuts through it along the asymptotic directions.

If every point of a surface is an elliptic point, κ is always positive and the surface is known as a *surface of positive curvature*. An ellipsoid is an example of positive curvature, and a sphere is a surface of constant positive curvature.

If all points of a surface are hyperbolic, then κ is always negative, and the surface is called a *surface of negative curvature*. A hyperbolic paraboloid is an example of a surface of negative curvature.

Finally, if all points of a surface are parabolic or planar, then $\kappa = 0$ and then $d^2 = 0$ and conversely. Thus, the only surfaces for which the total curvature is identically zero are the planes or the developable surfaces. A cone, for example, consists of only parabolic points.

A curve C on a surface S whose direction at each and every point is a principal direction is known as a *line of curvature*. If the surface is not a plane or a sphere, these are two principal directions at each point and they are mutually orthogonal. In this case, there are two families of lines of curvature, and they form an orthogonal system which satisfies the differential equation (4.7.53).

Every curve on a plane or a sphere is a line of curvature, because every direction at a point is a principal direction. It can be proved that a necessary and sufficient condition that the system of parametric curves consists of lines of curvature is that $F = 0$ and $f = 0$. We assume that the parametric curves are lines of curvature: $F = f = 0$ so that (4.7.48) reduces to

$$\kappa = \frac{e\,du^2 + g\,dv^2}{E\,du^2 + G\,dv^2}. \tag{4.7.57}$$

Since the directions of the parametric curves at a point P are the principal directions at a point P, the principal normal curvatures are determined by putting $dv = 0$ or $du = 0$ in (4.7.57). Consequently, if κ_1 is the principal normal curvature in the direction of the u-curve, and κ_2 that in the direction of the v-curve, then

$$\kappa_1 = \frac{e}{E}, \qquad \kappa_2 = \frac{g}{G}. \tag{4.7.58}$$

We next rewrite (4.7.57) in the form

$$\kappa = \kappa_1 \left(\frac{E\,du^2}{E\,du^2 + G\,dv^2} \right) + \kappa_2 \left(\frac{G\,dv^2}{E\,du^2 + G\,dv^2} \right). \qquad (4.7.59)$$

If α is the angle from the positive direction of the u-curve to the direction (dv/du), then the coefficient of κ_1 in (4.7.59) is $\cos^2\alpha$ and that of κ_2 is $\sin^2\alpha$. Consequently, equation (4.7.59) reduces to the *celebrated Euler equation* (4.7.26). In other words, the Euler equation states that the total curvature κ has the same value for two angles α which are negative to each other. In particular, at a hyperbolic point, the asymptotic directions are perpendicular if and only if $\kappa = 0$ when $\alpha = \pm\frac{\pi}{4}$, that is, if and only if the mean curvature $\frac{1}{2}(\kappa_1 + \kappa_2) = 0$. A surface, other than a plane, for which the $(\kappa_1 + \kappa_2)$ given by (4.7.54) is identically zero is known as a *minimal surface*. It is a surface of negative curvature at every point of which the asymptotic directions intersect at right angles. A *right helicoid* with parametric equations

$$x = u\cos v, \quad y = u\sin v, \quad z = av, \quad a \neq 0 \qquad (4.7.60)$$

is an example of a minimal surface. It can easily be verified that $E = 1$, $F = 0$, $G = u^2 + a^2$, $D^2 = u^2 + a^2$. Similarly, in this case, $e = 0$, $f = -a/D$, $g = 0$, $d^2 = -a^2/D^2$. Since $e = 0$ and $g = 0$, the asymptotic directions at a point P are given by (4.7.56), that is, they are given by $du = 0$ and $dv = 0$ at P. Thus, at every point, the asymptotic directions are mutually orthogonal and therefore, the right helicoid is a minimal surface. In this case, the differential equation (4.7.51) becomes

$$du^2 - (u^2 + a^2)dv^2 = 0. \qquad (4.7.61)$$

Thus, the equations of two families of lines of curvature on the helicoid are

$$du \mp \sqrt{u^2 + a^2}\,dv = 0. \qquad (4.7.62)$$

Integrating these equations gives the equations of the lines of curvature, that is,

$$\sinh^{-1}\left(\frac{u}{a}\right) - v = c_1, \qquad \sinh^{-1}\left(\frac{u}{a}\right) + v = c_2, \qquad (4.7.63)$$

where c_1 and c_2 are an arbitrary constants.

In view of (4.7.55), it turns out that $\kappa = -a^2/(u^2 + a^2)^2$. Thus, the total curvature of the helicoid is the same at all point of a circular helix, $u =$ constant and tends to zero as the radius of the circular cylinder containing the helix becomes infinite.

We conclude this section by adding the revolutionary approach to the subject of the modern foundations of geometry by Bernhard Riemann based on the work of Gauss and Euler. The Riemannian geometry was not just an extension of Gauss' differential geometry of surfaces in three-dimensional space. It dealt with n-dimensional intrinsic geometry for any space. His totally new treatment was concerned with n-dimensional space as a manifold where a point in the manifold is represented by assigning special values to n variable parameters, $x_1, x_2, \cdots, x_n$ so that the set of all such possible points constitute the n-dimensional manifold. These n variables are called the *coordinates of the manifold.* He introduced the distance between two generic points whose corresponding coordinates differ only by infinitesimal amounts so that the square of this distance is

$$ds^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\,dx_i dx_j, \tag{4.7.64}$$

where g_{ij} are functions of coordinates $x_1, x_2, \cdots, x_n$, $g_{ij} = g_{ji}$ and the right hand side of (4.7.64) is always positive for all possible values of dx_i. This expression for ds^2 is a simple generalization of the Euclidian distance (or metric)

$$ds^2 = \sum_{i=1}^{n} dx_i^2 = dx_1^2 + dx_2^2 + \cdots + dx_n^2. \tag{4.7.65}$$

It is important to point out that Riemann's curvature for the n-dimensional manifold reduces to Gauss' total curvature of a surface in three-dimensional space.

Historically, Euclid met with a serious difficulty by defining parallel lines as coplanar straight lines which do not intersect however far they be extended in either direction and by adopting his famous *Parallel* (or *Fifth*) *Postulate* (or *Axiom*) as a basic assumption. This can be stated in modern language as follows: "If a point P does not lie on a straight line ℓ in a plane, then in the plane there is exactly one line m passing through P parallel to the line ℓ". This was inconsistent in Euclidean geometry and could not be proved on the basis of the Euclid other nine axioms. This was a famous unsolved problem in mathematics for over 2000 years. It was Karl Friedrich Gauss who first reconfirmed that the Euclid Parallel Axiom cannot be proved from other axioms. So, this axiom was not only a genuine geometrical concern, but a fundamental physical problem for a long time. In order to revolutionize Euclidean and differential geometries and to establish their relationship with the physical world, Riemann first created a totally new n-dimensional geometry in 1854 which is now as the *Riemannian* (or *non-Euclidean elliptic*) *geometry.* It not only opened the door for the creation of non-Euclidean geometries, but also provided the mathematical framework for the Einstein theory of general relativity. Riemann is regarded primarily as the most modern mathematician, but he was deeply concerned with the physical spaces and the relationship of mathematics to the physical world. He first realized that Euclidean geometry is *not* the geometry of physical space and so, it became absolutely necessary to discover a totally new non-Euclidean elliptic geometry or the Riemannian geometry which includes Euclidean geometry as a limiting case. In 1830, Nicolai Ivanovich Lobachevsky (1793-1856) of Russia, and in 1832, János Bolyai (1802-1860) of Hungary independently discovered a totally new non-Euclidean hyperbolic geometry in which the Euclid Parallel Axiom simply does *not* hold.

4.8 Spherical Trigonometry

Spherical triangles and great circles play important roles in several subjects including astronomy, geodesy and navigation. Because of its major applications in quantitative astronomy, spherical trigonometry was studied before plane trigonometry by many including Hipparchus (second century B.C.), Claudius Ptolemy (second century A.D.) who wrote the most influential book known as the *Almagest* (or *Great Collection*) extending the work of Hipparchus. On the other hand, modern progress in the subject was made in the 18th century by Leonhard Euler and German mathematician, A. F. Möbius (1790-1868) and others. Euler made some major contributions to quantitative astronomy and spherical trigonometry.

In order to describe Euler's work on spherical trigonometry, we need some basic concepts, and notations of spherical triangles and great circles. The set of all points in space whose distance from a fixed point O are equal to a fixed distance r is called the *sphere* of *radius* r and *center* O. It can be proved that if a plane intersects a sphere in more than one point, the intersection is a circle. If a plane passes through the center of a sphere, its intersection with the sphere is a *great circle* of radius r. A circle of intersection of a sphere with a plane not passing through the center of the sphere is called a *small circle* of the sphere, and so, the radius of a small circle is less than the radius r of the sphere.

If two points A and A' of a sphere are opposite ends of a diameter, they are *antipodal* or A' is the *antipode* of A. Two great circles intersect in a pair of antipodal points and divide the surface of the sphere into four regions called *lunes*, as shown in Figure 4.25.

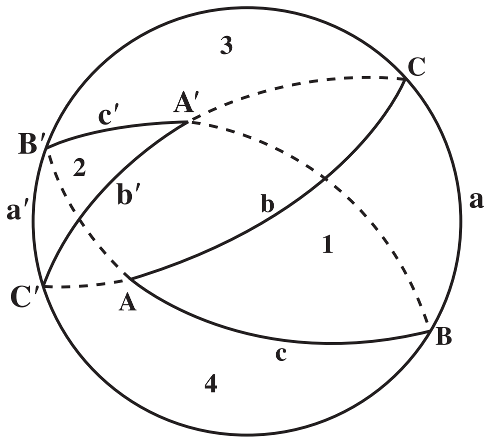


Fig. 4.25 A spherical triangle ABC and the four lunes.

Throughout this section, we denote a spherical triangle by ABC, its angles at the vertices A, B, C by A, B, C, respectively, and its sides by $a = \text{arc}\, BC$, $b = \text{arc}\, AC$ and $c = \text{arc}\, AB$, so that A, B, $C < \pi r$ and $0 < a + b + c < 2\pi r$.

The spherical triangle $A'B'C'$ is called the *polar spherical triangle*, as shown in Figure 4.25. Obviously, it has three angles A', B', C' and three sides a', b' and c'.

Euler Theorem 4.8.1. The area of a spherical triangle ABC on the unit sphere is

$$\triangle = A + B + C - \pi. \tag{4.8.1}$$

The Figure 4.25 shows that the great circles containing the sides of the spherical triangle ABC intersect pairwise in the vertices A, B and C and their corresponding antipodes A', B' and C'. The hemisphere in front of the great circle $BCB'C'$ is divided into four spherical triangles ABC, $AB'C'$, $AB'C$ and ABC' whose areas are denoted by $\triangle$, $\triangle_1$, $\triangle_2$, and $\triangle_3$. It follows from the Figure 4.25 that the spherical triangle $A'BC$ is congruent to the spherical triangle $AB'C'$ of area $\triangle_1$. In view of the fact that the area of the lune of angle A is $2A$, we obtain

$$\triangle + \triangle_1 = \text{area of the lune } A = 2A,$$
$$\triangle + \triangle_2 = \text{area of the lune } B = 2B,$$
$$\triangle + \triangle_3 = \text{area of the lune } C = 2C,$$

so that

$$2\triangle = 2A + 2B + 2C - (\triangle + \triangle_1 + \triangle_2 + \triangle_3) = 2A + 2B + 2C - 2\pi.$$

This proves the desired result.

More generally, the area of a spherical triangle ABC on a sphere of radius r is $(A + B + C - \pi)r^2$.

Since the surface area of the sphere of radius r is $4\pi r^2$, we obtain

$$0 < (A + B + C - \pi)r^2 < 4\pi r^2.$$

Or

$$\pi < A + B + C < 5\pi. \tag{4.8.2}$$

Euler Theorem 4.8.2. (The Law of Sines). In any spherical triangle ABC, the sine of the sides are proportional to the sine of the opposite angles. Or, in mathematical notations,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{4.8.3}$$

Proof. We consider a spherical triangle ABC and draw a great circular arc CD perpendicular to arc AB or arc AB extended. In Figure 4.26, CD lies inside or outside the spherical triangle ABC. In either case, it follows from the right-angled spherical triangle ACD and Napier's rules,

$$\sin\, h = \sin b\, \sin A. \tag{4.8.4}$$

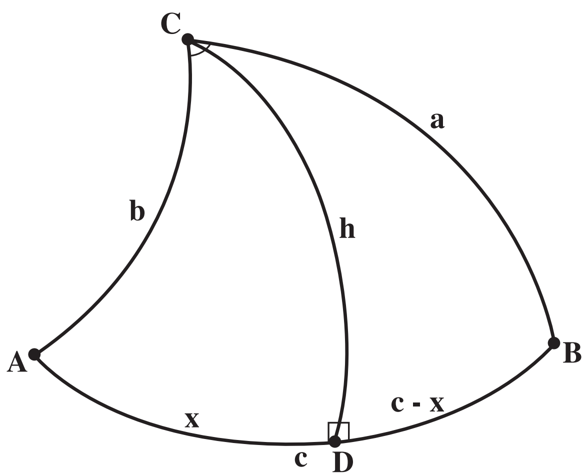


Fig. 4.26 A spherical triangle ABC with angles A, B and C, and sides a, b and c.

Similarly, from the right-angled spherical triangle BCD, we obtain

$$\sin h = \sin a \sin B. \tag{4.8.5}$$

Equating (4.8.4) and (4.8.5) gives

$$\sin a \sin B = \sin b \sin A.$$

Dividing both sides by $\sin A \sin B$ yields

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}. \tag{4.8.6}$$

Similarly, drawing a perpendicular from the vertex A to arc BC, we can show that

$$\frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{4.8.7}$$

Obviously, equations (4.8.6) and (4.8.7) are equivalent to the desired result (4.8.3).

Euler Theorem 4.8.3. (The Law of Cosines for sides). In any spherical triangle ABC,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{4.8.8}$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B, \tag{4.8.9}$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \tag{4.8.10}$$

Proof. Using Figure 4.26 with arc $AD = x$ and arc $DB = c - x$ and applying Navier's rules to right-angled spherical triangle BCD, we obtain

$$\cos b = \cos h \cos x, \quad b \neq 90°. \tag{4.8.11}$$

Similarly, from the right-angled spherical triangle ACD, we

$$\cos a = \cos h \cos(c - x). \tag{4.8.12}$$

Dividing (4.8.12) by (4.8.11) implies that, since $\cos b \neq 0$, we obtain

$$\frac{\cos a}{\cos b} = \frac{\cos(c - x)}{\cos x} = \cos c + \sin c \tan x.$$

Or,

$$\cos a = \cos b \cos c + \cos b \sin c \tan x. \tag{4.8.13}$$

It follows from the spherical triangle ACD and Navier's rules that

$$\cos A = \cot b \tan x.$$

Or,

$$\tan x = \cos A \tan b. \tag{4.8.14}$$

Substituting $\tan x$ from (4.8.14) in (4.8.13) gives the desired result (4.8.8).

Similarly, formulas (4.8.9) and (4.8.10) can be proved by drawing perpendiculars from the vertices A and B. In other words, formulas (4.8.9) and (4.8.10) can be derived from (4.8.8) by a cyclic permutation of the letters.

Euler Theorem 4.8.4. (The Law of Cosines for angles). In any spherical triangle,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a, \tag{4.8.15}$$

$$\cos B = -\cos C \cos A + \sin C \sin A \cos b, \tag{4.8.16}$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c. \tag{4.8.17}$$

Proof. We consider the spherical polar triangle $A'B'C'$ of the triangle ABC. We use the law of cosines for sides to $A'B'C'$ to obtain

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A. \tag{4.8.18}$$

Since $a' = (\pi - A)$, $b' = (\pi - B)$, $c' = (\pi - C)$, and $A' = (\pi - a)$, and since $\cos(\pi - \theta) = -\cos\theta$ and $\sin(\pi - \theta) = \sin\theta$, it follows from (4.8.18) that

$$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C\,(-\cos a).$$

Multiplying this result by (-1) gives the desired formula (4.8.15).

Similarly, formulas (4.8.16) and (4.8.17) can be derived.

Euler Theorem 4.8.5. (The half-angle formulas).

$$\sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}, \tag{4.8.19}$$

$$\sin\frac{B}{2} = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c \sin a}}, \tag{4.8.20}$$

$$\sin\frac{C}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}, \tag{4.8.21}$$

where $2s = (a + b + c)$.

Proof. We use the formula (4.8.8) for the law of cosine for sides as

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \tag{4.8.22}$$

and substitute it to the trigonometric identity

$$\begin{aligned} 2\sin^2\frac{A}{2} &= 1 - \cos A \\ &= 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ &= \frac{\cos(b-c) - \cos a}{\sin b \sin c} \\ &= \frac{2\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a-b+c)}{\sin b \sin c}. \end{aligned} \tag{4.8.23}$$

Using $s-a=\frac{1}{2}(-a+b+c)$, $s-b=\frac{1}{2}(a-b+c)$ and $s-c=\frac{1}{2}(a+b-c)$ in (4.8.23), we obtain the desired result (4.8.19).

Similarly, we obtain

$$\begin{aligned} 2\cos^2\frac{A}{2} &= 1+\cos A = 1+\frac{\cos a-\cos b\cos c}{\sin b\sin c} \\ &= \frac{\cos a-\cos(b+c)}{\sin b\sin c} \\ &= \frac{2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(-a+b+c)}{\sin b\sin c}. \end{aligned}$$

Consequently,

$$\cos\frac{A}{2}=\sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}. \tag{4.8.24}$$

Similarly,

$$\cos\frac{B}{2}=\sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}}, \qquad \cos\frac{C}{2}=\sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}. \tag{4.8.25}$$

Dividing (4.8.19) by (4.8.24) gives

$$\tan\frac{1}{2}A=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}. \tag{4.8.26}$$

Multiplying the numerator and denominator inside the radical sign in (4.8.26) by $\sin(s-a)$ and simplifying yields the result

$$\tan\frac{A}{2}=\frac{1}{\sin(s-a)}\cdot\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \tag{4.8.27}$$

Similarly, we can derive the following similar results

$$\tan\frac{B}{2}=\frac{1}{\sin(s-b)}\cdot\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \tag{4.8.28}$$

$$\tan\frac{C}{2}=\frac{1}{\sin(s-c)}\cdot\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}. \tag{4.8.29}$$

Chapter 5

Euler's Formula for Polyhedra, Topology and Graph Theory

> "In topology we are concerned with geometrical facts that do not even involve the concepts of a straight line or plane but only the continuous connectiveness between points of a figure."

David Hilbert

> "It often happens that understanding of the mathematical nature of an equation is impossible without a detailed understanding of its solution."

Freeman Dyson

5.1 Euler's Formula for Polyhedra

Although the study of geometry, in general and polyhedra, in particular held a central place in Greek geometry, it remained for Euler to discover a remarkable topological formula for simple polyhedra. In the eighteenth century, topology was considered the study of position and so, it was also known as *analysis situs* which dealt with the qualitative behavior of geometrical figures. Euler may be considered as the founding father of topology. He was universally known for the topological discovery in 1752 of the celebrated polyhedra formula

$$V - E + F = 2, \tag{5.1.1}$$

where V denotes the number of vertices, E the number of edges, and F the number of faces in a *simple regular polyhedron* (which is also called a *Platonic solid*). By a *polyhedron* is meant a solid whose surface consists of a number of polygonal faces.

On the basis of the Euler formula (5.1.1), it is easy to show that there are precisely 5 regular polyhedra (cube, tetrahedron, octahedron, dodecahedron and icosahedron). For suppose that a regular polyhedron has F faces, each of which is an n-sided regular polygon, and that r edges meet at each vertex. Counting edges by faces and vertices, we see that $nF = 2E$ for each edge belongs to two faces, and therefore is counted twice in the product nF. Further, since each edge has two vertices, $rV = 2E$. Thus, the formula (5.1.1) becomes

$$\frac{2E}{r} - E + \frac{2E}{n} = 2. \tag{5.1.2}$$

Or, equivalently,

$$\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E}. \tag{5.1.3}$$

We already know that $r \geq 3$ and $n \geq 3$, since a polygon must have at least three sides, and at least three sides must meet at each polyhedral angle. But r and n cannot both be greater than three, for then the left hand side of equation (5.1.3) could not exceed $\frac{1}{2}$ which is not possible for any positive number E. Thus, we have to find out what values r may have when $n = 3$ and what values n may have when $r = 3$. It turns out that the total number of polyhedra given by these two cases is precisely 5. When $n = 3$ equation (5.1.3) becomes

$$\frac{1}{r} = \frac{1}{6} + \frac{1}{E}, \tag{5.1.4}$$

r can therefore have values 3, 4, or 5 (6 or any number greater than 6 is not possible, because $\frac{1}{E}$ is always positive). For these values of $n = 3$ and $r = 3$, 4, 5, we find $E = 6$, 12, or 30, corresponding to the tetrahedron, octahedron, and icosahedron respectively. Similarly, for $r = 3$, equation (5.1.3) gives

$$\frac{1}{n} = \frac{1}{6} + \frac{1}{E}. \tag{5.1.5}$$

Thus, it follows from this equation that $n = 3$, 4, or 5, and $E = 6$, 12, or 30 respectively. These values correspond to the tetrahedron, cube (hexahedron) and dodecahedron respectively. Substituting these values for n, r, and E in $nF = 2E$ and $rV = 2E$, we obtain the number of vertices and faces of all five regular polyhedra: tetrahedron ($V = 4$, $E = 6$, $F = 4$), cube ($V = 8$, $E = 12$, $F = 6$), octahedron ($V = 6$, $E = 12$, $F = 8$), dodecahedron ($V = 20$, $E = 30$, $F = 12$) and icosahedron ($V = 12$, $E = 30$,

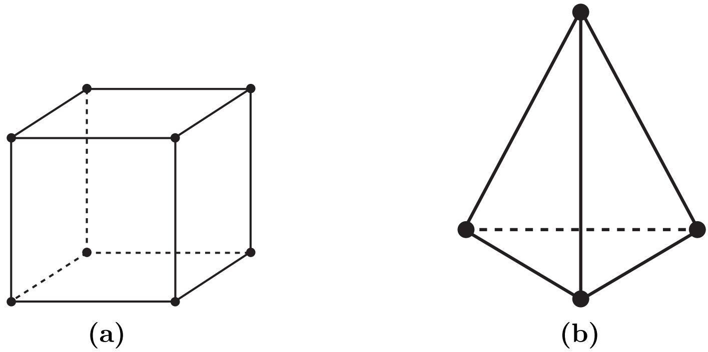


Fig. 5.1 (a) Cube ($V = 8$, $E = 12$ and $F = 6$), (b) Tetrahedron ($V = 4$, $E = 6$ and $F = 4$)

$F = 20$). The first two regular polyhedra - the cube and tetrahedron are shown in Figure 5.1 (a) and 5.1 (b).

If a given simple polyhedron is a hollow with a surface made of thin rubber, then we cut out one of the face of the hollow polyhedron so that the number of polygons will be one less than in the original polyhedron, since one face was removed. However, the resulting network of vertices and edges will contain the same number of vertices and edges as in the original polyhedron. Consequently, it turns out that for the network

$$V - E + F = 1. \tag{5.1.6}$$

This is also universally known as the *Euler formula*, where V is the number of vertices, E the number of edges, and F the number of faces of the given network. Mathematically, a *network* is a generalization of a graph, that is, a collection of vertices (or nodes) that are connected by edges (or links). A couple of simple examples of network is given in Figure 5.2.

The above classic work of Euler in the area of topology is now known as the *theory of graphs and networks* which constitutes finite geometry.

The Euler formula (5.1.1) holds for any simple polyhedron and does not hold for a non-simple polyhedron. But the range of validity of this formula goes far beyond the polyhedra of ordinary geometry, with their flat faces and straight edges. However, the proof of the Euler formula would apply equally well to a simple polyhedron with curved faces and edges, or to any subdivision of a surface of a sphere into regions bounded by curved arcs.

Modern studies of topology began with the Euler celebrated polyhedra formula. The modern idea of rigor in analysis started from a theorem of

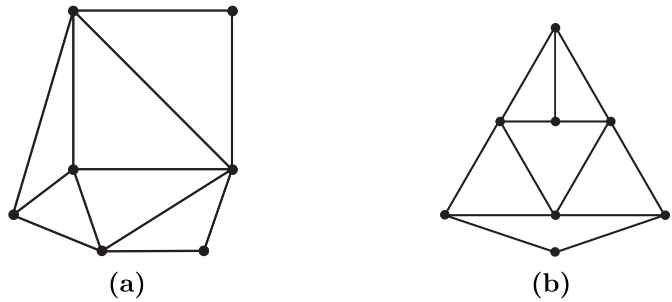


Fig. 5.2 Network (a) ($V = 7$, $E = 12$, and $F = 6$) and Network (b) ($V = 8$, $E = 13$ and $F = 6$).

Camille Jordan (1838-1922) which states that a simple closed curve C in the plane divides the plane into exactly two regions, an inside and an outside. This theorem, first stated in Jordan's famous book *Cours d'Analyse*, is obviously true for a circle or an ellipse, but it is not that evident for a complicated curve like twisted polygon. In fact, the Jordan curve theorem is quite easy to prove for the well-behaved curves such as polygons or curves with continuously turning tangents. It follows from the Jordan curve theorem that topology deals with providing rigorous proofs for many simple and obvious assertions. The renowned "*Four-Color Problem (or Conjecture)*" is a good example which states that four colors are sufficient to color any map on a plane so that areas with common boundaries are colored differently. In coloring a geographical map, it is necessary to use different colors to any two countries that have a portion of their boundary in common. The Four-Color Problem can be stated in the following mathematical theorem: For any subdivision of the plane into non-overlapping regions, it is always possible to mark the regions by one of the numbers 1, 2, 3, 4 in such a way that no two adjacent regions have the same number. The Four-Color Problem has indeed been proved for all maps containing less than 38 regions. So, it remains one of the great unsolved problem or conjecture in mathematics. A remarkable fact associated with the four-color problem is that for surfaces more complicated than the plane or the sphere the corresponding theorems have really been proved. Paradoxically, the analysis of more complicated geometrical surfaces seems in this respect to be easier than that of the simplest cases. Many simple but important topological facts occur in the study of two-dimensional surfaces. For example, when we compare the surface of a sphere with that of a torus, two surfaces are fundamentally different. On the sphere, as in the plane, every closed curve separates the surface into two parts. But on the torus there exist closed curves that do not separate into two parts. These facts suggest that a definition of *genus* of a surface as the largest number of non-intersecting simple closed curves that can be drawn on the surface without separating it. So the genus of the sphere is zero and that of the torus is one. The genus of a surface with two holes is two and that of a surface with g holes is g. The genus is, therefore, a topological property of a surface and remains the same if the surface is deformed.

In order to introduce the Euler characteristic of a surface, we consider a closed surface S of genus g that can be divided into a number of regions by marking a number of vertices on S and joining them by curved arcs. It can be proved that

$$V - E + F = 2 - 2g, \tag{5.1.7}$$

where V is the number of vertices, E the number of arcs, and and F is the number of regions. The number $\eta = 2-2g$ is called the *Euler characteristic* of a closed surface of genus g and thus, η depends only on the surface on which the map is drawn, and *not* on the map itself. Thus, the Euler convex polyhedra corresponds to maps on a sphere of genus zero so that the Euler characteristic, $\eta = 2$, whereas Cauchy's plane networks correspond to maps on a plane so that $\eta = 1$. It is relatively simpler to study the topological nature of surfaces by means of plane polygons with certain pairs of edges conceptually identified. The method of identification can be used to define three-dimensional closed manifolds similar to the two-dimensional closed surfaces. It is important to point out that the Euler's classic formula (5.1.1) has subsequently been generalized by Henri Poincaré in higher dimensions, and indeed, it is a special case of a general topological result. Since the surface of the regular polyhedra are all homeomorphic to the sphere, they have genus zero and the Euler characteristic 2.

Using arguments which were topological in a sense, Euler completely solved in 1736 the famous puzzle of the *Seven Bridges of the Prussian city of Königsberg* on the River Pregel where tributaries meet at an island as shown in Figure 5.3. The problem is to determine a route around the city so that one can cross all seven bridges once and only once. Many people made an attempt to devise such a route and had always failed. Euler first treated the problem from a mathematical point of view and proved that such a route is impossible. To demonstrate his approach, Euler observed that the Königsberg bridge-problem seems a suitable candidate and writes: "I have therefore decided to give here the method which I have found for

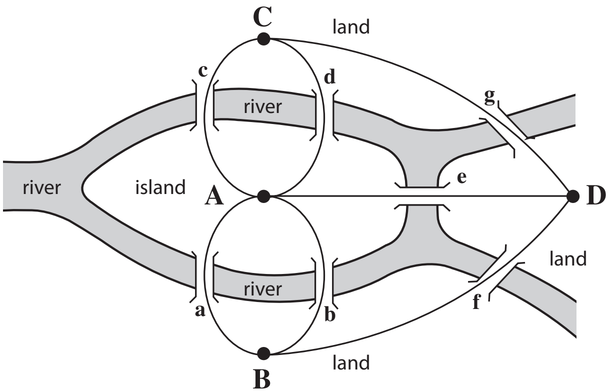


Fig. 5.3 The Königsberg Seven Bridges Problem (Ian Stewart, 2008).

solving this kind of problem, as an example of the geometry of position." Euler then describes the problem in Figure 5.3 with his notation as follows:

"From this, I have formulated the general problem: whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find whether or not it is possible to cross each bridge exactly once? ... My whole method relies on the particularly convenient way in which the crossing of a bridge can be represented. For this I use the capital letters A, B, C, D for each of the land areas separated by the river. If a traveler goes from A to B over bridge a or b, I write this as AB."

Euler's abstract formulation of this problem is shown by the Königsberg graph in Figure 5.4 (a) where land areas are the nodes (vertices) and edges are connecting bridges. It follows from the Euler formulation that it is *not* possible to devise such a route since all four nodes (representing land areas) in this case are of odd degree, where the *degree* of a node is the number of edges which meet at the node. In other words, Euler's graph of the problem as shown in Figure 5.4 (a) where the vertices C, A, B, and D representing the different land areas shows that a route is impossible. However, with an extra bridge added as shown in Figure 5.4 (b), one route $CgDeAcCdAaBfDhBbA$ is now possible, where CgD denotes the path that begins at vertex C and traverses edge g to arrive at the vertex D. Remarkably, Euler's treatment of the Königsberg bridge-problem led him to formulate a more general mathematical problem in graph theory which dealt with finding a path which contains each edge of the graph once and only once. Such is a path is known as the *Eulerian path*. Euler also proved

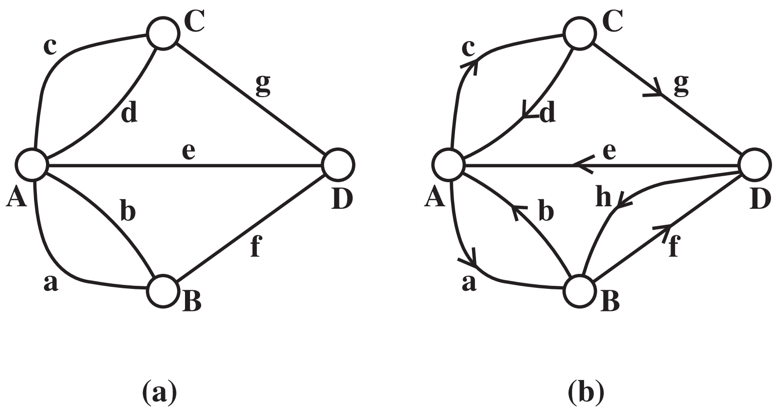


Fig. 5.4 The Eulerian Graph: (a) with Seven Bridges and (b) with Eight Bridges.

the existence of Eulerian paths in general graphs. After his remarkable discovery of graph theory in 1736, the last three centuries have produced major advances in mathematical theory of graphs and networks and their applications to a wide variety of subjects. These include, among others, physics, chemistry, engineering, business, electronics, computer science, sociology, psychology and transportation.

More precisely, topology is the study of those properties of geometric objects which remain unchanged under *bi-uniform and bi-continuous transformations*. Such transformations can be thought of as bending, stretching, twisting or compressing or any combination of these. It is assumed that the object being deformed or transformed is essentially elastic and capable of any degree of such manipulation. In his famous book on '*Anschauliche Geometrie*' (*Geometry and Imagination*), David Hilbert remarked: "In topology we are concerned with geometrical facts that do not even involve the concepts of straight line or plane but only the continuous connectiveness between points of a figure." Although some of the original ideas go back to Euler, it was really with the brilliant work of Henri Poincaré in 1895 that is universally known as the beginning of systematic attempts for developing topology as an independent mathematical discipline on its own merit.

In his letter to Goldback in 1757, Euler mentioned the problem of Knight's Tour on a chessboard in graph theory and combinatorics. A knight tour is a sequence of moves around the chessboard of the sixty-four squares which returns the knight to the square at which it began and visits no other square more than once. Such tours which include each of the sixty-four squares correspond to the Hamiltonian circuits. In 1759, Euler solved the problem using a graph in which the nodes represent the sixty-four squares. The Knight's Tour Problem is a special case of a general problem in graph theory, that is, whether it is possible to find a circuit, in a given graph, which passes through each vertex only once. Euler not only solved the Knight's tour problem on a standard chessboard, but also formulated the Knight's tour on a *non-standard chessboard* — one with n squares in each direction, instead of the usual eight squares. In graph theory, this is now known as *Hamilton's Tour Problem* after the name of the Irish mathematician. Sir William R. Hamilton who worked on such problems in the 1850s. Indeed, Euler solved the Knight's Tour problem as an early special case of the Hamilton's Tour Problem. Hamilton also considered another famous puzzle problem consisting of twenty vertices of a solid dodecahedron where each vertex represents a city in Europe. This problem was to find a route along edges which visited each city exactly once. The graph in Figure 5.5 represents a two-dimensional version of the Hamilton's puzzle where the nodes corresponds to the vertices of the dodecahedron and branches to the edges. In the language of graph theory, the problem is to solve Hamilton's puzzle by finding a Hamiltonian circuit in the graph.

In 1514, Albrecht Dürer's masterpiece painting, *Melencolia I*, there was a picture, called a *magic square* which was apparently developed for math-

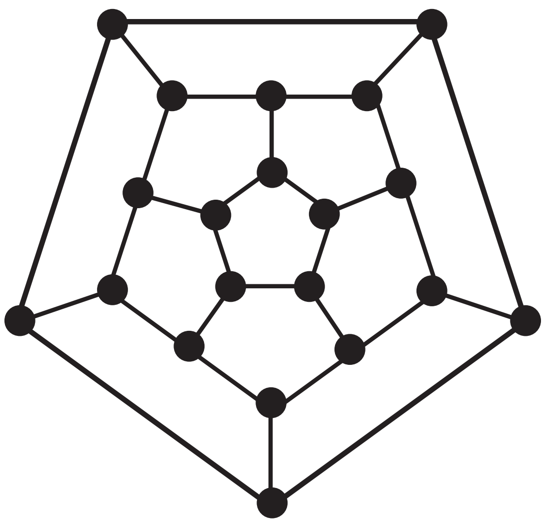


Fig. 5.5 The graph of Hamilton's puzzle.

ematical entertainment. Thus, the magic square is a 4×4 matrix M_4 of integers of the form

$$M_4 = \begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}.$$

All integers from 1 to 16 are used to formulate the 4×4 magic square M_4 with a special property which states that the sum of each row, column, and two diagonals is 34. Thus, an $n \times n$ matrix M_n is called a *magic square* if the sum of the elements in each row, each column, and both diagonals is the same. Clearly, the common sum of the rows, columns, and diagonals is a function of n, and is called the *weight* of M_n, defined by $wt(M_n)$

$$wt(M_n) = \frac{n(n^2+1)}{2}. \tag{5.1.8}$$

Furthermore, the $n \times n$ magic square matrix M_n contains each of the entries $1, 2, 3, \cdots, n^2$ exactly once.

Using a modern and sophisticated computer program, MATLAB designed especially for matrix computations, the 3×3 magic square matrix M_3, and the 5×5 magic square matrix M_5, are given below

$$M_3 = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad \text{and} \quad M_5 = \begin{bmatrix} 17 & 24 & 1 & 8 & 15 \\ 23 & 5 & 7 & 14 & 16 \\ 4 & 6 & 13 & 20 & 22 \\ 10 & 12 & 19 & 21 & 3 \\ 11 & 18 & 25 & 2 & 9 \end{bmatrix}.$$

Thus, it follows from the matrices M_3 and M_5 and formula (5.1.8) that

$$wt(M_3) = 15 \quad \text{and} \quad wt(M_5) = 65. \tag{5.1.9}$$

Many great mathematicians like Euler and the British mathematician, Arthur Cayley (1821-1895) found magic square puzzle very entertaining and worth studying from a mathematical point of view. There are also other mathematical puzzles of considerable interest. Among these are problems closely related with the theory of probability, map-coloring and the Euler *Knight Tour Problems*. In 1779, Euler spent a considerable amount of time to investigate the classical magic square from a mathematical point of view.

It is also equally important to point out that the problem of magic square has closely been related to another famous old problem, known as *Graeco–Latin Squares*. In response to a question posed to him by the Empress Catherine, Euler first considered the Graeco–Latin Squares in 1782 which dealt with 36 army officers, six each of the six different ranks and six different regiments so that they can be placed in a square such that exactly one officer in each rank and from each regiment appears in each row and column. Euler investigated this royal assignment mathematically and made a famous conjecture that a Graeco–Latin square of size n would never exist for any integer n of the form $(4k + 2)$ as he was not able to prove it. After 200 years later, Euler's conjecture was proved to be incorrect by several mathematicians including Bose and Shrikhande (1960), Bose et al. (1962), Klyve and Stemkoski (2006) of the twentieth century using different modern techniques from many areas of mathematicians and statistics, including graph theory, finite fields, projective geometry, block designs and modern computers. In his 1776 paper, Euler showed that Graeco–Latin Squares are closely related to magic squares and used Graeco–Latin squares of orders 3, 4 and 5 to construct magic squares. Since he was unable to construct Graeco–Latin Squares of order 6, Euler used a different method to construct magic squares of order 6. In his famous paper, Zhu Lie (1982) gave the most elegant disproof of the Euler conjecture using his own new method combined with the singular direct method. On the other hand, Stinson (1984) conclusively proved the 36-officer problem by using a transverse design, finite vector spaces and graph theory. It is evident from the above discussion that Euler's work and conjecture ranked among the most fertile problems in the history of mathematics. So, there are much more important and interesting problems in mathematics for the 21st century and still the subject of active research.

With the development of Georg Cantor's (1845-1918) theory of infinite sets and transfinite numbers, the set theory was firmly established as the foundation of mathematics, in general, and analysis, in particular. Once David Hilbert said: "Cantor created a paradise from which nobody shall expel us." The deeper study of set theory led to that of topology from the analytic and abstract point of view. The generalization of one of Cantor's most suggestive innovation in conceiving any geometrical configuration as a set of points in the Euclidean space, by René Maurice Fréchet (1878-1956) and Felix Hausdorff (1868-1942) and by others to abstract spaces led to a rapid development of analytic topology. Fréchet was responsible for the theory of metric spaces in topology. It is generally believed that "... topology became of the first-rate scientific importance in dynamical researches of Poincaré, particularly in connection with the problem of three-bodies attracting one another in space according to the Newtonian law of gravitation, for example the sun and two of its planets. It was a question of describing the families of possible orbits. Numerical calculation was too laborious and too slow to reveal the extremely intricate motions for more than a step at a time. A qualitative attack was indicated, and for this Poincaré (1895, 1900, 1904) created a major division of topology. He originated a rigorous combinatorial topology for space of any finite number of dimensions. Some of what he did has still to be surpassed." Poincaré not only discussed the set topology and algebraic topology, but also developed successfully the combinatorial methods to study invariant properties of complexes and to elucidate the full significance of the *Betti numbers*. He is the founding father of the dynamical systems and qualitative behavior of mathematics.

Considerable progress on topology was made during the first half of the twentieth century. Since the middle of 1950s, the major focus in the development of topology has undoubtedly been in the study of manifolds. Roughly, a manifold is a generalization of the idea of space or surface to any number of dimensions. So, simple examples of one-dimensional manifold are just curves (with the real line $\mathbb{R}$, as a special case), and two-dimensional manifolds are surfaces (with two-dimensional plane or manifold, $\mathbb{R}^2$, as a special case). In other words, a plane consists of all points uniquely determined by a pair of numbers (x, y). It is therefore a two-dimensional manifold. The space studied in three-dimensional analytical geometry may be considered as a three-dimensional manifold because each point is uniquely determined by three coordinates (x, y, z). In general, n numbers $(x_1, x_2, \cdots, x_n)$ are required to specify each point of an n-dimensional space, and so it is called an *n-dimensional manifold*. However, it is possible to think of many familiar kinds of manifolds which have nothing to do with space or geometry. It is possible to draw pictures of one or two-dimensional manifolds, however, manifolds of dimensions three or more cannot easily be illustrated. However, there are many new remarkable discoveries in the theory of manifolds of dimensions three, four or more during the second half of the twentieth century.

5.2 Graphs and Networks

Historically, Euler's original formulation of the Königsberg Seven-Bridge Problem in terms of graphs represented the start of the graph theory as it is known today. On the other hand, the Four-Color Problem can also be reformulated in terms of coloring the network so that the nodes (or vertices) of the network can be colored in such a way that any two nodes which are connected together must have different colors. If all networks can be colored using four different colors so can all maps and vice versa. Thus, the network formulation of the Four-Color Problem provides an alternative way of looking at it, and leads to the mathematical study of networks in finite geometry.

In modern mathematical language, a network is called *graph* which consists of a set of n points called *vertices* (or *nodes*) a_1, a_2, $\cdots$, a_n. The *lines* (or *curves*) joining the vertices are called *edges* (or *links*). A typical graph G of ten vertices, 12 straight edges, and one curved edge joining the vertex a_1 to vertex a_2 is shown in Figure 5.6 (a). Another graph G_1 of three vertices and four edges is shown in Figure 5.6 (b).

In the same graph G, there can be more than one line or curve joining a pair of vertices as in the Euler Seven-Bridge Problem. In graph G, there are 12 straight edges and one curved edge. For example, there is a straight edge and a curved edge joining the first vertex a_1 to the second vertex a_2 which can be represented by the straight edge a_{12} and a curved edge a_{12}. In order to be quite general, some vertices may be disconnected, such as the vertex, a_{10} and some edges may cross others, such as a_{69} and a_{78}.

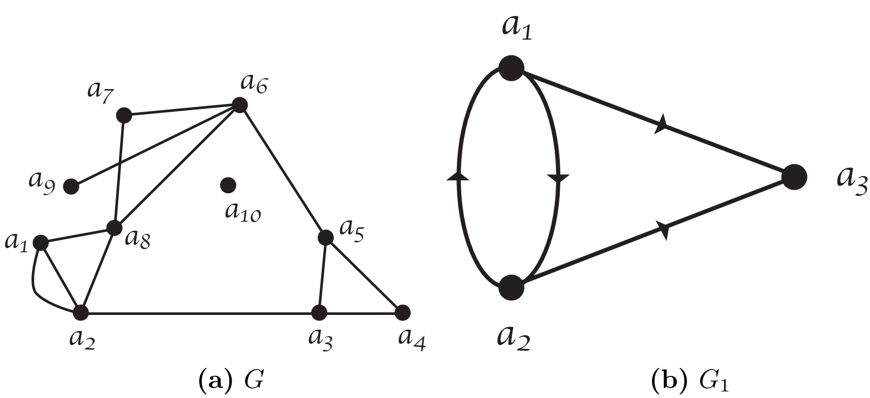


Fig. 5.6 (a) A graph G of ten vertices, (b) A graph G_1 of three vertices and four edges.

It can be shown that a graph G with n vertices can have at most $n(n-1)/2$, edges. For example, the graph G in Figure 5.6 (a) has 10 vertices without curved edge a_{12} so that the actual number of edges is 12 and the maximum numbers of straight edges is $10(10-1)/2 = 45$. The number of edges E can also be related to the *degree* of the vertices. In the graph G in Figure 5.6 (a), at the vertex a_3, there are three edges, so its degree is 3 which is usually denoted by $d_3 = 3$. At the vertex a_9, there is only one edge so that $d_9 = 1$. On the other hand, there is no edge at the isolated vertex a_{10}, and hence, $d_{10} = 0$. By counting all edges and noting each edge joins two vertices (G has no curved edges), it follows that the sum of all the degrees of the vertices is equal to *twice* the total number of edges E so that

$$\sum_{n=1}^{10} d_n = 2E = 2 \cdot 12. \tag{5.2.1}$$

It is important to point out that it is the degree of the vertices of the Eulerian graph that provide the hint of understanding the Euler problem of crossing the Königsberg seven bridges. Since the degree of every vertex of the Eulerian graph as shown in Figure 5.4 (a) is odd, the solution of Euler's seven-bridge problem is impossible,that is, there is no route which starts at C, A, B, or D, and ends at C, A, B, or D and crosses each bridge only once. However, with one extra curved edge added between D and B as in Figure 5.4 (b), the solution of the Euler problem is possible, because the degrees of B and D are even. So it is possible to start at C and end at A, or

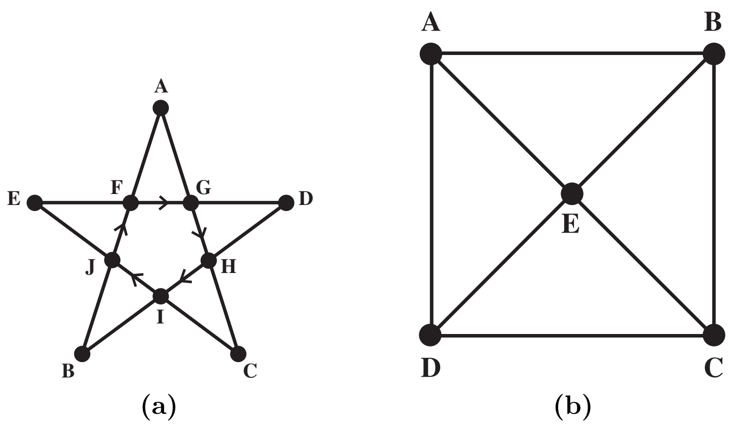


Fig. 5.7 (a) A star graph, (b) A graph with five vertices and eight edges.

vice versa. Hence, the solution of the Euler seven-bridge problem is possible. In solving the seven-bridge problem, Euler did much more than merely solve the famous puzzle. He described the existence of certain fundamental properties of geometrical figures in no way dependent on, or related to, shape or size. These properties are functions only of the general position of the vertices and edges of a figure. The problem of seven bridges is a representative of a class of problems, some dating back to antiquity. They demonstrate the difficulties of mentally grasping the real geometric properties of all but the simplest figures. For example, the historic star symbol, known from the Hindus to the Pythagorean, drawn in Figure 5.7 (a) can be transversed with a single pencil stroke as $AGHCIJEFGDHIBJFA$.

Euler's solution of the seven-bridge problem explained why the graph in Figure 5.7 (b) cannot be traversed with a single pencil stroke, for there are 5 vertices, 4 of which are odd degrees, and hence, two journeys are required to transverse the graph — a single journey will not suffice.

The pentagon graph as shown in Figure 5.8 is far more complicated in appearance, but can be transversed with a single stroke. Starting at the vertex A, the journey would successively pass through vertices $ABCDEFGBHIDJFAGHCIJEA$.

In general, the matrix notation a_{ij} is often used in graph theory to denote the edges from the ith vertex to the jth vertex and a_{ji} to denote the number of edges from the jth vertex to the ith vertex. Sometimes, the edges in a graph represent connections that only operate in one direction so that

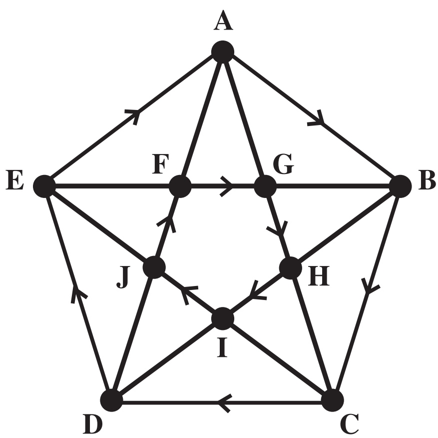


Fig. 5.8 A pentagon graph.

arrows are used to indicate these directions. A graph of this type is called a *directed graph* (or *digraph* for short). It is convenient to store a large graph in a computer in the matrix form and then use the modern computer program, MATLAB to perform matrix computations. So, the matrix algebra, first discovered by Arthur Cayley in 1858, has become an effective tool to study the graph theory as well as the network theory. More remarkable is that, sixty seven years after Cayley's ingenious discovery, Werner Heisenberg (1901-1961) in 1925 recognized in the algebra of matrices exactly the method he needed for his revolutionary work in quantum mechanics.

An *adjacency matrix* $A = [a_{ij}]$ of a graph G is defined by

$$a_{ij} = \begin{cases} r, & \text{if vertices } i \text{ and } j \text{ are joined by } r \text{ edge,} \\ 0, & \text{otherwise.} \end{cases}$$

The adjacency matrix A of the graph G in Figure 5.9 is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

An *adjacency matrix* $A = [a_{ij}]$ of a digraph G in which there is exactly

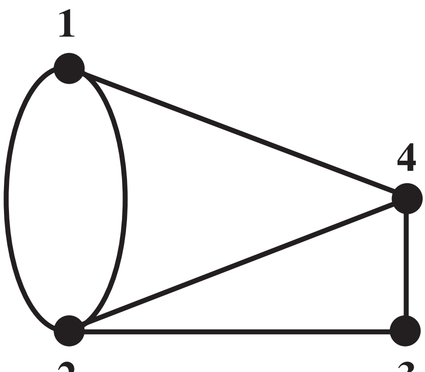
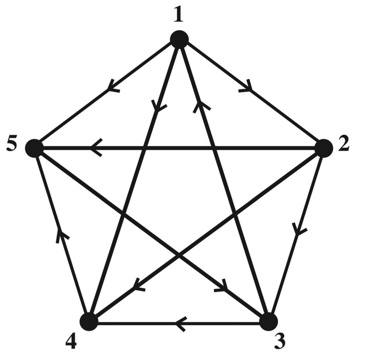


Fig. 5.9 Graph G of four vertices and six edges.



(a)

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(b)

Fig. 5.10 (a) Diagraph G of a tournament of five players 1 to 5, (b) Its adjacency matrix A.

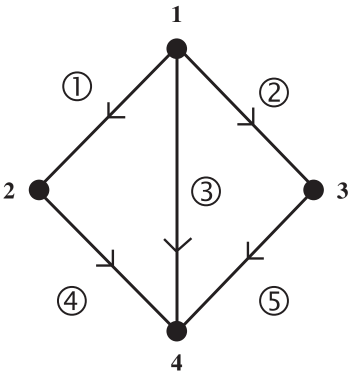
one directed edge between every pair of vertices is defined by

$$a_{ij} = \begin{cases} 1, & \text{if vertices } i \text{ and } j \text{ are joined by an arrow from } i \text{ to } j, \\ 0, & \text{otherwise.} \end{cases}$$

The adjacency matrix A of the digraph G in Figure 5.10 (a) is given in Figure 5.10 (b).

The *incidence matrix* $A = [a_{mn}]$ associated with a digraph G consisting of n vertices connected by m edges is an $m \times n$ matrix whose rows are indexed by the edges and whose columns are indexed by the vertices. If edge r starts at vertex i and ends at vertex j, then the rth row of the incidence matrix will have $+1$ in its (r, i) entry and -1 in its (r, j) entry; all other entries in the row are zero. So, $+1$ corresponds to the outgoing vertex at which the edge starts and -1 the incoming vertex at which it ends. The incidence matrix A of a diagraph G in Figure 5.11 (a) consisting of 5 edges joined at 4 different vertices is of size 5×4 is given by in Figure 5.11 (b)

Hence, the first row of the incidence matrix A says that the first edge starts at vertex 1 and ends at vertex 2. Similarly, the second row states that the second edge goes from vertex 1 to vertex 3 and so on. Clearly, it is easy to construct an incidence matrix from a given digraph, and conversely, it is easy to construct any digraph from its given incidence matrix. It is noted that the incidence matrix provides important geometric and quantitative information of the given digraph. In particular, its kernel and cokernel have topological significance. For example, the kernel of the incidence



(a)

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

(b)

Fig. 5.11 (a) A simple digraph, (b) Its incidence matrix A.

matrix A associated with the digraph in Figure 5.11 is spanned by the single vector $\mathbf{u} = (1, 1, 1, 1)^T$ and represents the fact that sum of the elements of any given row of the matrix A is zero. This observation is true in general for connected graphs. More precisely, if A is the incidence matrix for a connected digraph, then $ker A$ is one-dimensional with basis $\mathbf{u} = (1, 1, \cdots, 1)^T$. Moreover, if A is the incidence matrix for a connected digraph with n vertices, then the rank of A is $n - 1$.

In modern terminology, it is important to mention that an Eulerian path is a particular case of general class of graphs, known as the *complete graphs*. A *complete graph* with $n\,(> 1)$ vertices is a simple graph in which every vertex is joined to every other vertex, and it is denoted by K_n. In other words, a graph of n vertices is complete if it has exactly $n(n-1)/2$ edges. The graphs of K_2, K_3, K_4 and K_5 are drawn in Figure 5.12. Since there is exactly one edge between pair of nodes in a complete graph, the graph K_n has n vertices of degree $(n-1)$. Using arguments similar to those

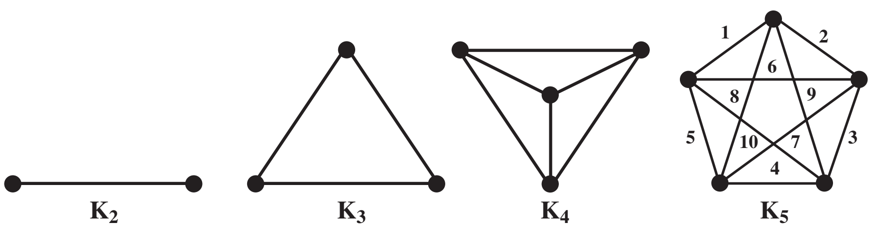


Fig. 5.12 The complete graphs K_n, $n = 2$, 3, 4 and 5.

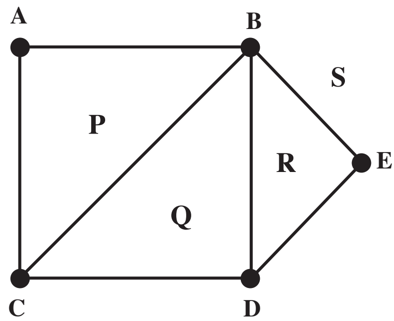


Fig. 5.13 A planar graph.

of Euler, Louis Poinsot (1777-1859) showed that an Eulerian path in K_n is impossible when $n = 4, 6, 8, \cdots$ because in these cases there are more than two vertices with odd degree. He also discovered an ingenious method for construction an Eulerian path in K_n for odd n.

On the other hand, a graph is called *connected* if, for any two distinct vertices, there is an edge joining them together. Obviously, a connected graph does not contain isolated vertices. In other words, each of its vertices has at least one edge. There is another kind of graph which is abundantly utilized in designing electrical circuits so that no wires cross. Such a graph, with edges crossing or meeting except at vertices is known as *planar graph* or equivalently, about maps on the plane. The graphs K_2, K_3 and K_4 shown in Figure 5.12 are planar, but the graph K_5 and all succeeding ones are not planar. Another planar graph with five vertices, seven edges and four faces is shown in Figure 5.13.

In 1813, Cauchy's generalization of the Euler formula (5.1.1) is a theorem about planar graphs. He considered the result of allowing extra vertices and edges inside a polyhedron so that it is, in fact, decomposed into number p of separate polyhedra and obtained the formula

$$V - E + F = p + 1 \tag{5.2.2}$$

which reduces to the Euler formula (5.1.1) for polyhedra when $p = 1$, and to (5.1.6) for networks when $p = 0$.

However, the most common practical example of a graph which *cannot* be represented by a planar graph is the energy delivery system graph consisting of three services, water (W), gas (G), and electricity (E), to three houses A, B, and C as shown in Figure 5.14 (a). This graph has no plane drawing.

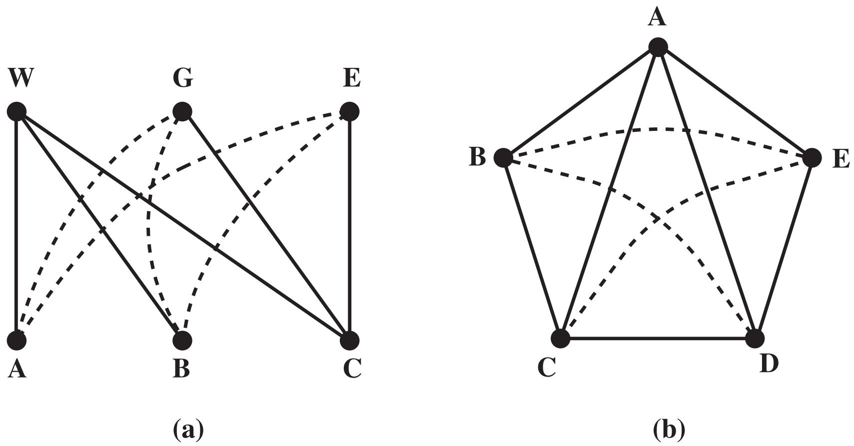


Fig. 5.14 (a) Bipartite graph $K_{3,3}$ and (b) Complete graph K_5.

The graph in Figure 5.14 (a) is a good example of another graph which is known as *bipartite graph* in which one set of vertices may be connected to another set of vertices, but *not* to vertices in the same set. If every vertex in one set is connected by one edge to every vertex in the other set, then it is called a *complete bipartite graph*. If two sets have m and n vertices respectively and each vertex in the first set is joined to each vertex in the second set so that there are exactly mn edges, then the notation $K_{m,n}$ is used to denote the complete bipartite graph. For example, the graphs $K_{1,2}$, $K_{2,2}$ and $K_{2,3}$ are drawn in Figure 5.15 with circular and square dots representing the two sets of vertices. Obviously, these graphs are planar as the plane drawing of them have no crossings at all. However, the graphs $K_{3,3}$ and K_5 in Figure 5.14 (a) and 5.14 (b) are important examples of *non-planar graphs*.

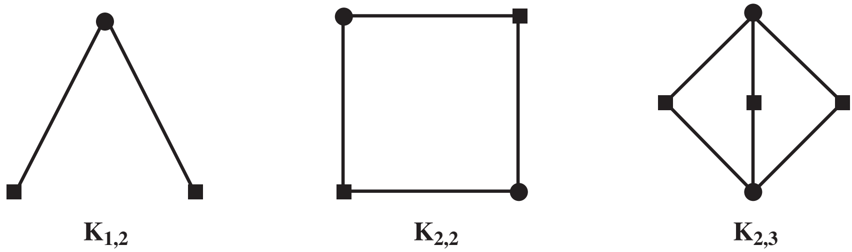


Fig. 5.15 Complete planar bipartite graphs.

In general, there is a relation between the numbers of vertices, edges and faces of a planar graph. In a plane drawing of a graph, the plane is divided into regions called faces and one face always represents the region external to the graph. The planar graph in Figure 5.13 has *four* faces P, Q, R and the external faces, S. Evidently, Euler's formula (5.1.1) holds for any planar graph. In 1930, the Polish mathematician Kazimierz Kuratowski (1896-1980) proved a remarkable theorem that every non-planar graph has a subgraph homeomorphic to either K_5 and $K_{3,3}$.

It turns out that the structure of a graph is related to that of one-dimensional complex in topology. Some geometrical figure F consisting of distinct vertices and curves (either arcs of circles or line segments) is called a *geometric realization of a graph* G, if there exists a one-to-one correspondence between the vertices of F and those of G, and also between the curves of F and edges of G so that the corresponding curves and edges connect the corresponding vertices. So, there is a fundamental question whether any graph G can be realized in Euclidean space. If it can, then does there exist a value n such that any graph admits realization in Euclidean space of dimension n? What is then the minimum value of n? There is a well-known theorem in topology which provides answers to these questions. This theorem states that every finite graph G can be realized in *three-dimensional* Euclidean space. However, it is proved in topology that graphs K_5 and $K_{3,3}$ do not admit realization on the plane.

This section 5.2 is essentially devoted to a brief introduction to the theory of graphs and networks with many examples from the real world. Included are some of the basic properties of graphs and networks for some understanding of the macroscopic behavior of real physical systems. We now close this section by stating some important and modern applications of graph theory or network problems from transportation to telecommunications. Graphs or networks are effectively used as powerful tools in industrial, electrical and civil engineering, communication networks in the planning of business and industry. Graph theory and combinatorics can be used to understand the changes that occur in many large and complex scientific, technical and medical systems. With the advent of fast large computers, large scale complex optimization problems can be modeled in terms of graphs or networks and then solved by algorithms available in graph theory. Many large and more complex combinatorial problems dealing with the possible arrangements of situations of various different kinds, and computing the number and properties of such arrangements can be formulated in terms of networks.

Historically, the 1736 marked the Euler's original discovery of the solution of the Königsberg Seven Bridge problem in terms of graphs which represented the start of the graph theory. It was a remarkable coincidence that a Hungarian mathematician, Dénes König (1884-1944) published the first comprehensive treatise on graph theory, *Theorie der endlichen und unendlichen Graphen* in 1936 which marked the 200th anniversary of Euler's discovery. It was Euler who first laid the foundation of graph theory which had become a major branch of mathematics in its own right since 1936. Indeed, graph theory is now considered one of the most active and useful research areas in modern pure and applied mathematics.

Chapter 6

Euler's Contributions to Calculus and Analysis

> "Read Euler, read Euler, he is the master of us all."
>
> *P. S. Laplace*

> "The study of Euler's work will remain the best school for different fields of mathematics and nothing else can replace it."
>
> *Carl Friedrich Gauss*

> "Euler was the most prolific mathematician in history and the major figure in the development of analysis in the eighteenth century."
>
> *Victor Katz*

6.1 Introduction

Although Newton and Leibniz were universally recognized for the independent discovery of calculus, it was the Bernoulli brothers, Jakob (1654-1705) and Johann (1667-1748) who did considerable significant work to build the subject of calculus, in general and infinite series, in particular. Based on the greatest achievement of Newton, Leibniz and the Bernoulli brothers, Euler published his first two-volume masterpiece treatise on mathematical analysis entitled, *Introductio in analysin infinitorum* (*Introduction to the analysis of the infinite*) in 1748. This work is essentially concerned with the infinite processess of analysis: the expansion of functions in infinite series, infinite products, continued fractions, and summation of a wide variety of algebraic, trigonometric and hyperbolic functions. He established the idea of a function as the most fundamental concept in mathematical analysis. He defined it very broadly as: "A function of a variable quantity is an analytical expression composed in whatever way of that variable quantity and of numbers and constant quantities." He also introduced a function of several variables and proved some of its basic properties. Even though the concepts of limit and continuity were not rigorously established until the nineteenth century, he distinguished between continuous and discontinuous functions, between single-valued and multiple-valued functions, between explicit and implicit functions, and between algebraic and transcendental functions. He developed a fairly complete but non-rigorous theory of differentiation and integration in his *Institutiones Calculi Differentials* (*Foundations of Differential Calculus*) published in 1755. He introduced the definition of derivative as the "ratio of two vanishing increments," and then obtained the value of $\left(\frac{\delta y}{\delta x}\right)$ when δx is very small.

Euler first introduced the expansion of function in the form

$$f\,(x+\delta x) = f(x) + \left(\frac{df}{dx}\right)\delta x + O\left((\delta x)^2\right). \tag{6.1.1}$$

Or, equivalently,

$$\frac{df}{dx} = \frac{f\,(x+\delta x) - f(x)}{\delta x} + O\,(\delta x)\,. \tag{6.1.2}$$

Replacing the derivative by the difference is universally known as the *Euler method* to numerically solve ordinary differential equations. It was the above relation (6.1.2) that provided Euler his celebrated entrée into the world of numerical analysis.

Subsequently, he published a comprehensive three-volume textbook on integral calculus entitled *Institutiones Calculi Integralis* in 1768-1770. These volumes not only represented a comprehensive analytical treatment of integral calculus and analysis of the eighteenth century, but also contained a large number of his own discoveries. He also developed the techniques of integration in many different directions including the explicit evaluation of integrals of rational functions with or without trigonometric functions as factors. He also first discovered the so-called Eulerian integrals of the first and the second kind which are also known as the beta and the gamma functions, respectively. In addition, Euler made some major contributions to elliptic integrals as natural generalizations of the inverse circular functions. He used elliptic integrals to compute arclengths of the ellipse, hyperbola and lemniscate. In general, elliptic integrals play a significant role in a wide variety of applications to problems in geometry, number theory, and physics. The study of the motion of a simple pendulum also requires the use of elliptic integrals. The legacy of Euler's work in elliptic integrals led to the development of the theory of elliptic functions and elliptic curves.

6.2 Euler's Work on Calculus

Euler believed that functions can be represented by infinite series and infinite products, and introduced new notations, such as, e, π, $\log_e x$, $\cos x$, $\sin x$, and $f(x)$ for a function of x. In his *Introductio*, Euler defined the exponential and logarithmic functions as limits:

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \tag{6.2.1}$$

$$\ln x = \log_e x = \lim_{n\to\infty} n\left(x^{\frac{1}{n}}-1\right). \tag{6.2.2}$$

Evidently, it follows from (6.2.1) that e^x has the infinite series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \infty. \tag{6.2.3}$$

When $x = 1$, (6.2.3) gives the Euler number e which is a transcendental number of value 2.71828183. Putting $x = \pm 1$ in (6.2.1) leads to results for e and $\frac{1}{e}$ in the forms

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \qquad \frac{1}{e} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n. \tag{6.2.4}$$

On the other hand, the function defined by equation (6.2.2) is known as the *natural* (or *Napierian*) *logarithm* of x. It is often written as $\ln x$ to indicate that the logarithm is taken to the natural base of e, that is, $\ln x \equiv \log_e x$. Clearly, the above two equations (6.2.1) and (6.2.2) define two most important functions in mathematics: if $y = e^x$, then $x = \ln y$ and vice-versa. So, there is a simple inverse relationship between y and x.

Both Newton and Nicholas Mercator (1620-1687) independently expanded $f(x) = (1+x)^{-1}$ in the form

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \tag{6.2.5}$$

and then integrated term-by-term to obtain

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots. \tag{6.2.6}$$

This is a much more convenient means of calculating logarithms which were the artificial numbers of John Napier, the area under the rectangular hyperbola and sum of an infinite series. At the beginning of the seventeenth century, Napier used Fermat's approach to quadrature (or integration) to calculate the area under the hyperbola $y = x^{-1}$, and showed that the arithematic-geometric relationship is the characteristic property of the logarithms. While passing over many other individual contributions, Euler unified several approaches to logarithms which appeared in his great textbook on algebra *Complete Introduction to Algebra* of 1770. He also successfully clarified the controversy between Leibniz and Bernoulli regarding the logarithms of negative and imaginary numbers of introducing the multivaluedness of the logarithms.

Euler first recognized an amazing connection between logarithms and harmonic series which led his celebrated discovery of the Euler universal constant γ as follows. He substituted $x = \frac{1}{n}$ in (6.2.6) to obtain

$$\ln\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots . \tag{6.2.7}$$

Or, equivalently,

$$\frac{1}{n} = \ln\left(\frac{1+n}{n}\right) + \frac{1}{2n^2} - \frac{1}{3n^3} + \cdots . \tag{6.2.8}$$

For sufficiently large n $(n \to \infty)$

$$\frac{1}{n} = \ln\left(\frac{1+n}{n}\right) + O\left(\frac{1}{n^2}\right). \tag{6.2.9}$$

Putting $n = 1, 2, 3, \cdots$ in (6.2.8) gives

$$\begin{array}{lllll}
\frac{1}{1} = & \ln 2 & + \frac{1}{2} & - \frac{1}{3} & + \cdots \\
\frac{1}{2} = & \ln\left(\frac{3}{2}\right) & + \frac{1}{2.2^2} & - \frac{1}{3.2^3} & + \cdots \\
\frac{1}{3} = & \ln\left(\frac{4}{3}\right) & + \frac{1}{2.3^2} & - \frac{1}{3.2^5} & + \cdots \\
\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \\
\frac{1}{n} = & \ln\left(\frac{n+1}{n}\right) & + \frac{1}{2.n^2} & - \frac{1}{3.n^3} & + \cdots
\end{array}$$

Euler added these results column-wise to obtain the harmonic series and the sum of logarithms plus a numerical constant so that

$$\sum_{k=1}^{n} \frac{1}{k} \sim \log(n+1) + 0.57721567\cdots . \tag{6.2.10}$$

In the limit as $n \to \infty$, the Euler universal constant γ is now defined by

$$\gamma = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(1+n)\right]. \tag{6.2.11}$$

This is equivalent to

$$\begin{aligned}
\gamma &= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln n - \ln(1+n) + \ln n\right] \\
&= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right] - \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) \\
&= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right].
\end{aligned} \tag{6.2.12}$$

It follows from (6.2.6) that

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right). \tag{6.2.13}$$

Substituting $x = 1$ in series for $\ln(1-x)$ shows the divergence of harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \ln(1-1) = \ln(0) = \infty. \tag{6.2.14}$$

When $x = \frac{1}{3}$, (6.2.13) gives

$$\ln 2 = 2\left(\frac{1}{3} + \frac{1}{81} + \frac{1}{1215} + \cdots\right) = 0.69314. \tag{6.2.15}$$

In his *Introductio*, Euler first introduced the trigonometric and hyperbolic functions and developed their analytic treatment in a remarkable way. He also introduced the mathematical symbol i for $\sqrt{-1}$ so that any complex number z can be written as $x + iy$, where x and y are real numbers. He then replaced x by ix or $-ix$ in (6.2.3) to obtain

$$\exp(\pm ix) = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \pm i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right). \tag{6.2.16ab}$$

Thus, two bracketed expressions in (6.2.16ab) represent the infinite power series for the trigonometric functions $\cos x$ and $\sin x$, respectively, so that (6.2.16ab) can be written as

$$e^{ix} = \cos x + i \sin x \quad \text{and} \quad e^{-ix} = \cos x - i \sin x. \tag{6.2.17ab}$$

Consequently, these results lead to the most beautiful and remarkable *Euler formulas* in mathematics:

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \quad \text{and} \quad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right). \tag{6.2.18ab}$$

He also established the famous formulas

$$(\cos x \pm i \sin x)^n = \cos nx \pm i \sin nx \tag{6.2.19}$$

and generalized them for all real values of n. These formulas were known to Abraham de Moivre (1667-1754) and are universally known as *de Moivre's formulas*, but he never stated them explicitly. Evidently, they follow immediately from the equality

$$[\exp(ix)]^n = \exp(inx). \tag{6.2.20}$$

Euler provided the formulation of the addition theorem for cosine and sine functions using the formula

$$(\cos x + i \sin x)(\cos y + i \sin y) = \cos(x+y) + i \sin(x+y). \tag{6.2.21}$$

Equating the real and imaginary parts of (6.2.21) gives the familiar addition theorems for the cosine and sine functions.

In 1746, Euler gave a pleasant surprise to the entire mathematical community by showing an imaginary power of an imaginary number, i is a real number. For example, $i^i = \exp(-\pi/2)$. He later proved that there are infinitely many values of i^i. It is remarkable that Euler discovered two elegant and most beautiful formulas in mathematics

$$e^{i\pi} + 1 = 0 \quad \text{and} \quad e^{2\pi i} - 1 = 0. \tag{6.2.22ab}$$

These formulas relate six most fundamental constants e, i, π, 0, 1 and -1 in mathematics.

In his long and magnificent article entitled 'Recherches sur les racines imaginaires des equations' published in 1749, Euler introduced a new method of finding roots of complex numbers and then expressed the nth roots of a complex number $z = x + iy$ in the form

$$(x+iy)^{\frac{1}{n}} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i \sin\left(\frac{\theta + 2\pi k}{n}\right)\right], \tag{6.2.23}$$

where $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, and $k = 0, 1, 2, \cdots, (n-1)$.

This led him to determine a beautiful short cut for finding nth roots of unity $(x^n = 1)$ given by

$$\omega_k = \exp\left(\frac{2\pi i k}{n}\right) = \cos\left(\frac{2\pi k}{n}\right) + i \sin\left(\frac{2\pi k}{n}\right), \quad k = 0, 1, 2, \cdots, (n-1), \tag{6.2.24}$$

where the sum $\omega_0 + \omega_1 + \omega_2 + \cdots + \omega_{n-1} = 0$. Geometrically, the nth roots of unity represent the vertices of a regular polygon of n sides inscribed in the unit circle ($|z| = r = 1$) and one of those vertices is at $z = 1$.

In his 1755 *Institutiones Calculi differentialis*, Euler considered a general polynomial function

$$y = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \cdots \tag{6.2.25}$$

for real values of x, and proved that the values of x which make the function $y(x)$ a maximum or minimum are the roots of

$$\frac{dy}{dx} = nx^{n-1} - A(n-1)x^{n-2} + \cdots = 0. \tag{6.2.26}$$

He also investigated that the equation $F(x) = 0$ of degree n will have n real roots and the derived equation $F'(x) = 0$ of degree $(n-1)$ will have $(n-1)$ real roots. He proved that between two consecutive real roots of the equation $F(x) = 0$, there is always a real root of the equation $F'(x) = 0$. In his study of polynomial equations, Euler made no reference of Michel Rolle's (1652-1719) work on similar equations. Whether or not he had read Rolle's work is not known. In 1690, Rolle published his famous *Traité d' algebre* (*Treatise of algebra*) in which he described his *method of cascades* for finding roots of polynomial equations. Originated as an unexpected theorem in algebra, Rolle made the first statement of his great theorem in 1690, known as *Rolle's theorem* in real analysis, which can be stated in modern notations as follows. If $f(x)$ is a continuous function in the closed interval $[a, b]$ and differentiable in the open interval (a, b), then there exists a point $x = c$ in (a, b) such that $f'(c) = 0$. At that time, the Taylor series had not yet been discovered and calculus was its very early stage of development. Indeed, Rolle's theorem or its equivalent form was published several times during the eighteenth century because it has been regarded as one of the fundamental theorems in real analysis. However, some mathematics historians claimed that a version Rolle's theorem was first stated by the Indian mathematician, Bhaskara, in the twelfth century without a formal proof. A proof of the theorem had to wait until centuries later when Rolle in 1691 adopted the methods of the differential calculus. Subsequently, the proof of Rolle's theorem followed from the mean value theorem (or Taylor theorem) in differential calculus:

$$\frac{f(b) - f(a)}{b - a} = f'(c), \qquad a < c < b. \tag{6.2.27}$$

Geometrically, this means that the secants have the same slope as the tangent for some point c in (a, b) of the function $y = f(x)$.

On the other hand, Rolle's theorem can also be derived from the mean value theorem for integrals

$$\int_a^b f(x)dx = (b-a)f(c), \qquad a < c < b \tag{6.2.28}$$

when $f(x)$ is replaced by $f'(x)$ with $f(b) = f(a)$.

Euler derived many formulas for the derivatives of the rational, algebraic, trigonometric and transcendental functions. Based on an article entitled Euler's Mathematical Notebooks by Knobloch published in the 2007 English edition, of Euler and Modern Science, it may be appropriate to mention Euler's major contributions to a large variety of indefinite and definite integrals involving algebraic rational functions, trigonometric functions and exponential functions. Euler's listed an incredible number of integrals with or without evaluation. These notebooks also provide a vivid idea of Euler's numerous research activity and research creativity of many diverse areas including astronomy, geography, mechanics, physics, pure and applied mathematics.

Euler also defined a homogeneous function $u(x, y)$ of degree n by the condition

$$u(tx, ty) = t^n u(x, y) \tag{6.2.29}$$

for any real t. He then proved that

$$x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = nu(x, y). \tag{6.2.30}$$

This is known as the *Euler theorem* for the homogeneous function $u(x, y)$ of degree n.

The condition (6.2.29) is equivalent to

$$u(x, y) = x^n\, g\left(\frac{y}{x}\right) \quad \text{or} \quad y^n\, h\left(\frac{x}{y}\right). \tag{6.2.31}$$

We differentiate (6.2.31) partially with respect to x and y and then make simple algebraic manipulation to obtain the Euler theorem (6.2.30).

In general, all homogeneous function $u\,(x_1, x_2, \cdots, x_m)$ of degree n in the variables x_1, x_2, $\cdots$, x_m are characterized by the condition

$$u\,(tx_1, tx_2, \cdots, tx_m) = t^n\, u\,(x_1, x_2, \cdots, x_m) \tag{6.2.32}$$

which is true for real values of t. If we set $t = \frac{1}{x_m}$, we have

$$u\,(x_1, x_2, \cdots, x_m) = x_m^n\, v\left(\frac{x_1}{x_m}, \frac{x_2}{x_m}, \cdots, \frac{x_{m-1}}{x_m}, 1\right) \tag{6.2.33}$$

which is equivalent to

$$u(x_1, x_2, \cdots, x_m) = x_m^n \, v\left(\frac{x_1}{x_m}, \frac{x_2}{x_m}, \cdots, \frac{x_{m-1}}{x_m}\right), \tag{6.2.34}$$

where v is a function of $(m-1)$ variables which satisfies the above condition of homogenity. Thus, (6.2.34) represents the totality of homogeneous function of degree n. A proof similar to that of homogeneous function of two variables can be adopted to prove the Euler theorem for the homogeneous function u of degree n in m variables in the form

$$x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2} + \cdots + x_m \frac{\partial u}{\partial x_m} = nu. \tag{6.2.35}$$

Although Newton first introduced polynomial equations $f(x, y) = 0$ in x and y to describe geometrical properties of plane curves, he made no attempt to publish partial derivatives of $f(x, y)$ with respect to x and y. James Bernoulli first used partial derivatives in his work on isoperimetric problems. On the other hand, Nicholas Bernoulli (1687-1759) utilized partial derivatives for finding orthogonal trajectories of plane curves which was published in *Acta Eruditorum* in 1720. Indeed, Euler, Clairaut and d'Alembert developed the theory of partial derivatives to create the calculus of functions of several independent variables and the theory of partial differential equations. Clairaut discovered the total differential du or df from $u = f(x, y)$ in terms of partial derivatives in the form

$$du = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy, \tag{6.2.36}$$

and then derived the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ for the exact differential of the expression $M(x, y)dx + N(x, y)dy$. In general, if $u = f(x_1, x_2, \cdots, x_n)$, the total differential df is

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots + \frac{\partial f}{\partial x_n} dx_n. \tag{6.2.37}$$

Multiple integrals appeared in the first part of the eighteenth century in the work of Euler who formulated the attraction of an elliptical lamina of thickness δc on a point directly over the center and distance c by means of the double integral

$$(\delta c) \iint \frac{c \, dx \, dy}{(c^2 + x^2 + y^2)^{3/2}} \tag{6.2.38}$$

taken over the ellipse $\left(x^2/a^2\right) + \left(y^2/b^2\right) = 1$. He also evaluated this double integral in 1738 by repeated integration with respect to y and then used infinite series of the resulting integrand as a function of x. Subsequently, Euler developed a fairly general method of evaluation of a double integral over a bounded domain by repeated integration.

In a series of papers, Euler introduced partial derivatives of functions $u = f(x, y, z, \cdots)$ of two or more independent variables. In one of his 1734 papers, he proved, for $u = f(x, y)$, that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}. \tag{6.2.39}$$

In his other papers written from 1748 to 1766, Euler treated change of variables, inversion of partial derivatives, and functional determinants. Almost simultaneously, d'Alembert generalized the calculus of partial derivatives in his works of 1744 and 1745 on analytical dynamics. The very success of the geometrical approach to calculus led to the widespread shift of emphasis on the greater power and versatility of calculus. The most striking feature of the eighteenth century mathematics was to build the logical foundations of the calculus by creating a new and vast subject of *mathematical analysis* as used by Euler and d'Alembert. It is a delight to quote d'Alembert's views in 1743 as follows: "Up to the present... more concern has been given to enlarging the building than to illuminating the entrance, to raising it higher than to giving proper strength to the foundations."

6.3 Euler and Elliptic Integrals

Euler's major interest in elliptic integrals and elliptic functions goes back to his early years with Johann Bernoulli. While he was at the Berlin Academy, on December 23, 1751, Euler received two-volume work of C. G. Fagnano entitled *Produzioni Matematiche*, published in 1750 for his formal review. This work contained the formula for the duplication of the arclength of lemniscate whose polar coordinate equation is $r^2 = a^2 \cos 2\theta$ and whose rectangular coordinate equation is $\left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2\right)$. Euler was tremendously inspired by this work and helped create a new area of algebraic functions and their duplication formula. In fact, the name elliptic integral originated from the problem of rectification of elliptic arcs. C. G. Jacobi announced the date of December 23, 1751 as the date of birth of elliptic integrals (and of elliptic functions). Indeed, Fagnano discovered a simple and remarkable solution of the rectification of the lemniscate, and he was so proud of his achievement that he left instructions to inscribe a lemniscate on his grave. In 1716, Fagnano proved that two arcs of any given ellipse may be determined in an infinite number of ways so that their difference is equal to a segment of a given straight line. Being impressed by the original work of Fagnano and others about the importance and richness of the theory of elliptic integrals, Euler began his systematic study of elliptic integrals with their geometrical and physical applications. His great paper of 1757 in which he proved the famous addition theorem for elliptic integrals which established a new major subject of independent interest in mathematical analysis. Indeed, among his landmark discoveries are celebrated addition and multiplication theorems for elliptic integrals and their new applications. Euler's significant achievement received a praise from André Weil's (1984) statement: "With characteristic generosity Euler never ceased to acknowledge his indebtedness to Fagnano; but surely none but Euler would have seen in Fagnano's isolated result the germ of a new branch of analysis. His first contribution was to extend Fagnano's duplication formula for the lemniscate to a general multiplication formula ...." Jean d'Alembert in Paris made some cordial mathematical correspondence with Euler in Berlin on elliptic integrals and the vibrating string problem governed by the wave equation over many years, and then d'Alembert made some important contributions to certain transformations of elliptic integrals with applications. Although there was a strong professional disagreement between d'Alembert and Euler in 1757 on mathematical ideas and problems, especially, on the vibrating string problem, Euler praised d'Alembert's contributions to integral calculus and elliptic integrals. All these early works concerned with the evaluation of certain integrals involved in special problems with a very little interest in the systematic investigation of general properties of these integrals may be regarded as the first new epoch in the history of development of the theory of elliptic functions. However, Euler's major work on elliptic functions was solely motivated by applications which began with his study of *elastica*, a curve described by a thin elastic rod compressed at the ends which was discovered by Bernoulli brothers.

In 1768, J. L. Lagrange solved some Euler's problems of elliptic integrals by elegant methods. In 1771, J. Landen (1719-1790) became very successful in finding that the hyperbolas can, in general, be rectified by means two ellipses. In this epoch, elliptic integrals were investigated systematically by A. M. Legendre and C. G. Jacobi who used rapidly convergent elliptic theta functions and discovered the Jacobian sn and cn functions $w = sn\ z$ and $w = cn\ z$ as new generalizations of trigonometric sine and cosine functions.

In order to understand Euler's famous work on addition and multiplication theorems for elliptic integrals, it may be appropriate to consider an elementary problem, that is, to find a function $z = h(x, y)$ such that the following integral formula

$$\int_a^x f(t)dt + \int_a^y f(t)dt = \int_a^z f(t)dt \tag{6.3.1}$$

holds for a given function f and constant a.

For example, if $a = 1$, $f(t) = t^{-1}$ and

$$L(x) = \int_1^x \frac{1}{t}\, dt = \ln x, \tag{6.3.2}$$

then

$$L(x) + L(y) = L(z) = L(xy), \tag{6.3.3}$$

where $z = h(x, y) = xy$ is the product of two real numbers x and y.

Similarly, if

$$s(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}} = \sin^{-1} x, \tag{6.3.4}$$

and

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} + \int_0^y \frac{dt}{\sqrt{1-t^2}} = \int_0^z \frac{dt}{\sqrt{1-t^2}} \tag{6.3.5}$$

or

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1} z \tag{6.3.6}$$

hold, we substitute $x = \sin A$ and $y = \sin B$ in (6.3.6) to obtain

$$z = \sin(A+B) = \sin A \cos B + \cos A \sin B = x\sqrt{1-y^2} + y\sqrt{1-x^2} = h(x, y). \tag{6.3.7}$$

Conversely, if $z = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, then differentiating partially with respect to x and y gives

$$\sqrt{1-x^2}\, \frac{\partial z}{\partial x} = \sqrt{1-y^2}\, \frac{\partial z}{\partial y}. \tag{6.3.8}$$

It follows from (6.3.6) that

$$A + B = \sin^{-1} z = \sin^{-1} (\sin A \cos B + \cos A \sin B)\,. \tag{6.3.9}$$

Or

$$\sin(A+B) = \sin A \cos B + \cos A \sin B. \tag{6.3.10}$$

This is the familiar addition theorem for the sine function.

In order to obtain the 1781 addition theorem of Fagnano, we consider the equation of lemniscate $r^2 = a^2 \cos 2\theta$ which is $r^{*2} = \cos 2\theta$, where

$r^* = (r/a)$. Dropping the asterisk, the equation of the lemniscate becomes $r^2 = \cos 2\theta$.

Thus, $rdr = -\sin 2\theta d\theta$ and

$$\left(\frac{dS}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = \cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta} = \sec 2\theta.$$

Consequently,

$$dS = \sqrt{sec\ 2\theta}\ d\theta = \frac{d\theta}{\left(1 - 2\sin^2\theta\right)^{\frac{1}{2}}}.$$

Substituting $t = \tan\theta$, $dt = \left(1 + t^2\right) d\theta$ gives $dS = (1 - t^4)^{-\frac{1}{2}} dt$.

The arclength of the lemniscate curve is

$$S(r) = \int_0^r \frac{dt}{\sqrt{1 - t^4}} \tag{6.3.11}$$

between the origin and a point on the curve. Fagnano obtained the *duplication formula*

$$2S(r) = S(R), \qquad R = \frac{2r\sqrt{1 - r^4}}{1 + r^4}. \tag{6.3.12}$$

This gives a prescription of how to double the lemniscate using a compass and ruler.

Inspired by this work of Fagnano, Euler discovered the addition theorem for the elliptic integrals. If

$$\int_0^x \frac{dt}{\sqrt{1 - t^4}} + \int_0^y \frac{dt}{\sqrt{1 - t^4}} = \int_0^z \frac{dt}{\sqrt{1 - t^4}}, \tag{6.3.13}$$

then

$$z = h(x, y) = \frac{x\sqrt{1 - y^4} + y\sqrt{1 - x^4}}{1 + x^2y^2} \tag{6.3.14}$$

is called the celebrated *Euler addition theorem*, where $z = h(x, y)$ is an algebraic symmetric function. When $x = y = r$, (6.3.14) reduces to the Fagnano duplication formula (6.3.12).

It is interesting to note that z satisfies the quadratic equation

$$z^2\left(1 + x^2y^2\right) - 2zx\sqrt{1 - y^4} + \left(x^2 - y^2\right) = 0. \tag{6.3.15}$$

Using (6.3.13) and (6.3.15), Euler proved that if

$$\int_0^y \frac{dt}{\sqrt{1 - t^4}} = n\int_0^x \frac{dt}{\sqrt{1 - t^4}} \tag{6.3.16}$$

holds, then y is an algebraic function of x. This result is called the *Euler multiplication theorem* for the elliptic integral $S(x)$ given by (6.3.11). From this result the complete integral of equation (6.3.16) can be obtained.

From these results, Euler was able to obtain what is now known as the *addition theorem for elliptic integral* of the first kind of the form

$$\int_0^x \frac{dt}{\sqrt{R(t)}} \tag{6.3.17}$$

where

$$R(t) = 1 + m\,t^2 + n\,t^4. \tag{6.3.18}$$

If

$$\int_0^x \frac{dt}{\sqrt{R(t)}} + \int_0^y \frac{dt}{\sqrt{R(t)}} = \int_0^z \frac{dt}{\sqrt{R(t)}} \tag{6.3.19}$$

holds, then $z = h(x, y)$ given by

$$z = \frac{x\sqrt{R(y)} + y\sqrt{R(x)}}{1 - n\,x^2y^2} \tag{6.3.20}$$

is the celebrated *Euler addition theorem*. When $m = -1$ and $n = 0$, (6.3.19) and (6.3.20) reduce to (6.3.5) and (6.3.7) respectively. On the other hand, when $m = 0$ and $n = -1$, (6.3.19) and (6.3.20) reduce to (6.3.13) and (6.3.14) respectively.

It is noted that

$$\sqrt{R(x)}\,\frac{\partial z}{\partial x} = \frac{1}{(1 - n\,x^2y^2)^2}\Big[\left(1 + n\,x^2y^2\right)\,\sqrt{R(x)}\,\sqrt{R(y)} + 2nxy\left(y^2 + \frac{m}{2}x^2y^2 + x^2\right) + mxy\Big]. \tag{6.3.21}$$

When $n = 0$ and $m = -1$, this reduces to

$$\sqrt{R(x)}\,\frac{\partial z}{\partial x} = \sqrt{R(x)}\,\sqrt{R(y)} - xy, \qquad R(x) = (1 - x^2). \tag{6.3.22}$$

This is in complete agreement with the partial derivatives of (6.3.7) with respect to x and y and then with (6.3.8).

Euler was also interested in finding the complete integral of

$$\frac{m\,dx}{\sqrt{1 - x^4}} = \frac{n\,dy}{\sqrt{1 - y^4}}, \tag{6.3.23}$$

where (m/n) is rational and this result represents the problem of finding two arcs of a lemniscate that have this ratio to each other. He was also convinced that (6.3.23) has a complete integral which can be represented by a certain algebraic equation in x and y when (m/n) is rational. In 1753, Euler discovered many addition formulas for elliptic integrals which are usually referred to *Euler's addition theorems*.

It is more remarkable to mention here the discovery of Gauss' *lemniscate functions* in 1796 when he was only nineteen years old. He was interested in the inverse function $r = r(S)$ of the elliptic integral (6.3.11). A simple differentiation of (6.3.11) gives

$$\frac{dS}{dr} = \frac{1}{\sqrt{1-r^4}}, \qquad -1 < r < 1. \tag{6.3.24}$$

When $\frac{dS}{dr} > 0$, the function $S(r)$ in $-1 < r < 1$ has an inverse function which was denoted by $r = s\ell\, S$, $-\omega < S < \omega$ and is called the *Gauss lemniscate sine function.* In particular, the quantity

$$\omega = \int_0^1 \frac{dt}{\sqrt{1-t^4}} \tag{6.3.25}$$

is simply the half arclength of the lemniscate curve. Gauss also introduced the *lemniscate cosine* function by the relation

$$c\,\ell\, S = s\ell\,(\omega - S), \tag{6.3.26}$$

and then proved the famous identity

$$s\ell^2 S + c\,\ell^2 S + s\ell^2 S\, c\,\ell^2 S = 1. \tag{6.3.27}$$

This is the generalization of the familiar trigonometric identity $\sin^2 s + \cos^2 s = 1$, where $x = r = \sin s$ defined by (6.3.4), and

$$\omega = \int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}, \quad \text{and} \quad \cos s = \sin\left(\frac{\pi}{2} - s\right). \tag{6.3.28ab}$$

Although Gauss introduced the lemniscate functions $r = s\ell\, S$ and $r = c\,\ell\, S$ for real values of S, but the most remarkable fact is that he extended these functions to complex values of S by writing (6.3.11) formally as

$$\int_0^{ir} \frac{dt}{\sqrt{1-t^4}} = i\int_0^r \frac{du}{\sqrt{1-u^4}}, \tag{6.3.29}$$

so that $S(ir) = i\,S(r)$. This led Gauss to the definition

$$s\ell\,(i\,S) = i\, s\ell\,(S) \qquad \text{for all } S \in \mathbb{R}. \tag{6.3.30}$$

With the aid of this definition and the addition theorem, he then defined $s\ell(S)$ for complex numbers S and derived two fundamental periodicity properties for all complex S as

$$s\ell\,(S + 4\omega) = s\ell\,(S) \quad \text{and} \quad s\ell\,(S + 4i\omega) = s\ell\,(S). \tag{6.3.31}$$

Unlike the trigonometric sine and cosine functions with a single real period 2π, the Gauss lemniscate functions are *doubly periodic meromorphic functions* with one real period 4ω and another purely imaginary period $4i\omega$. These functions are called the *Gauss elliptic functions*.

In 1757, Euler studied the deflexion $y(x)$ of a thin elastic vertical column of uniform cross-section and length ℓ when a constant vertical compressive force (or load) P is applied to its top. The bending moment $M(x)$ at any point at a point x along the column is proportional to the curvature κ of the elastic column so that

$$M(x) = EI\kappa = EI\frac{y''}{\left(1+y'^2\right)^{3/2}} \approx EIy'', \tag{6.3.32}$$

provided the slope of the deflexion y' is small, where E is a constant Young's modulus of elasticity of the column and I is the moment of inertia of the cross-section of the column and the product EI is called the *flexural rigidity* of the column. The differential equation of the deflexion $y(x)$ is

$$EI\,\frac{d^2y}{dx^2} = -Py. \tag{6.3.33}$$

Noting $\frac{dy}{dx} = \tan\psi$, $\frac{dy}{ds} = \sin\psi$, $\frac{dx}{ds} = \cos\psi$ and $ds^2 = dx^2 + dy^2$, it follows that $\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan\psi) = \sec^2\psi \cdot \frac{d\psi}{dx} = \sec\psi \cdot \frac{d\psi}{ds}$. Consequently, the Euler differential equation (6.3.32) becomes

$$EI\,\frac{d\psi}{ds} = -Py. \tag{6.3.34}$$

Differentiating this equation with respect to s and using $\frac{dy}{ds} = \sin\psi$ gives

$$\frac{d^2\psi}{ds^2} = -a^2\sin\psi, \qquad a^2 = \frac{P}{EI}. \tag{6.3.35}$$

Integrating this equation once yields

$$\frac{1}{2}\left(\frac{d\psi}{ds}\right)^2 = a^2\cos\psi + C, \tag{6.3.36}$$

where C is an integrating constant. At the upper end of the column, the bending moment is zero so that $\psi = \alpha$ and $\frac{d\psi}{ds} = 0$ and $C = -a^2\cos\alpha$. Thus, equation (6.3.36) reduces to

$$\left(\frac{d\psi}{ds}\right)^2 = 2a^2\left(\cos\psi - \cos\alpha\right). \tag{6.3.37}$$

Or, equivalently,

$$ds = -\frac{d\psi}{\sqrt{2}\,a\left(\cos\psi - \cos\alpha\right)^{\frac{1}{2}}}, \tag{6.3.38}$$

where the positive square root is not admissible because $\frac{d\psi}{ds} < 0$. Equation (6.3.38) represents the intrinsic equation of the curve formed by the column. However, this equation cannot be integrated to obtain an explicit form $s = f(\psi)$. Since $dy = \sin\psi\, ds$, equation (6.3.38) represents the deflection of the top of the column given by

$$\begin{aligned} y(\alpha) &= \frac{1}{a\sqrt{2}} \int_0^\alpha \frac{\sin\psi}{(\cos\psi - \cos\alpha)^{\frac{1}{2}}} d\psi = \frac{\sqrt{2}}{a} \left[(\cos\psi - \cos\alpha)^{\frac{1}{2}} \right]_\alpha^0 \\ &= \frac{\sqrt{2}}{a} (1 - \cos\alpha)^{\frac{1}{2}} = \frac{2}{a}\, \sin\frac{\alpha}{2}. \end{aligned} \tag{6.3.39}$$

Euler also solved the problem of a simple pendulum which consists of a light inextensible string of length ℓ to which a particle of mass m is attached at the lower end with the upper end fixed. In describing the motion of the simple pendulum in a vertical circle of radius ℓ and arclength $s = \ell\theta$, where θ is the angular displacement from the vertical, the angular acceleration is $\ddot{s} = \ell\ddot{\theta}$. According to Newton's second law, $m\ddot{s} = m\ell\ddot{\theta} = -mg\sin\theta$. Thus, the equation of motion of the pendulum is given by

$$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\, \sin\theta = -\omega^2 \sin\theta, \tag{6.3.40}$$

g is the acceleration due to gravity and $\omega^2 = g/\ell$. Integrating (6.3.40) once gives

$$\left(\frac{d\theta}{dt} \right)^2 = 2\omega^2 \cos\theta + C, \tag{6.3.41}$$

where C is an integrating constant and at $t = 0$, $\theta = \alpha$ is the initial angular displacement and $\dot{\theta} = 0$ so that $C = -2\omega^2 \cos\alpha$. Consequently,

$$\left(\frac{d\theta}{dt} \right)^2 = 2\omega^2 (\cos\theta - \cos\alpha) = 4\omega^2 \left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2} \right). \tag{6.3.42}$$

Integrating again gives

$$2\omega = \int_0^t dt = \int_0^\theta \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}} = \int_0^u \frac{2\, du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \tag{6.3.43}$$

which is obtained by substituting $\sin\frac{\theta}{2} = u \sin\frac{\alpha}{2}$ with $k = \sin\frac{\alpha}{2}$.

Consequently,

$$\omega t = \int_0^u \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}. \tag{6.3.44}$$

This represents the famous Jacobi elliptic integral of the first kind satisfied by the Jacobi *sn* function so that

$$u = sn\,(\omega t, k), \tag{6.3.45}$$

and

$$\sin\frac{\theta}{2} = k\ sn\,(\omega t, k), \quad \cos\frac{\theta}{2} = dn\,(\omega t, k), \tag{6.3.46}$$

where $dn\,(z, k)$ is the Jacobi elliptic *dn* function. Thus, the coordinates of the particle is given in terms of time t as

$$(\ell\sin\theta,\ \ell\cos\theta) = \left[2k\ell\ sn\,(\omega t, k)\ dn\,(\omega t, k), \quad \ell\{2dn^2(\omega t, k) - 1\}\right]. \tag{6.3.47}$$

When $\theta = \alpha$, $u = 1$, and the periodic time T for a complete oscillation of the pendulum is given by

$$T = 4\omega \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} = 4\omega \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \quad (u\sin\phi) \tag{6.3.48}$$

$$= 4\omega \int_0^{\pi/2} \left(1 + \frac{1}{2}k^2\sin^2\phi + \frac{1.3}{2.4}\sin^4\phi + \cdots\right) d\phi$$

$$= 2\pi\omega \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \cdots\right]. \tag{6.3.49}$$

It follows from (6.3.48) that the periodic time T has the exact form

$$T = 4\omega\, K(k) \tag{6.3.50}$$

where $K(k)$ is called the *complete elliptic integral of the first kind* defined by the integral

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2}\, F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right), \tag{6.3.51}$$

which has been expressed in terms of the *hypergeometric function* $F(a, b, c; x)$. More details about properties of $K(k)$ can be found in Dutta and Debnath (1965).

Since $sn\,(z, k)$ is a periodic function of z, the motion of a simple pendulum is periodic with period $4\omega\, K(k)$.

For a compound pendulum consisting of a rigid body oscillating about a horizontal axis rigidly connected with the body under gravity, the corresponding equation of motion is given by

$$I\,\frac{d^2\theta}{dt^2} = -mgh\sin\theta, \tag{6.3.52}$$

where I is the moment of inertia of the body about the axis of rotation and h is the distance of the center of gravity from the axis of rotation, and m is the mass of the body.

Thus, the equation of motion (6.3.52) for a compound pendulum is

$$\left(\frac{I}{mh}\right)\frac{d^2\theta}{dt^2} = -g\sin\theta. \tag{6.3.53}$$

Evidently, the motion of a compound pendulum is identical with that of a simple pendulum of length $\ell = (I/mh)$ and frequency $\omega = \sqrt{g/\ell} = \sqrt{\frac{mgh}{I}}$.

Euler's more work on elliptic functions and elliptic integrals is discussed in Section 11.3 of Chapter 11. His brilliant work stimulated tremendous interest to many greatest mathematicians including C. F. Gauss, Christof Gudermann (1798-1852), N. H. Abel, Evaristé Galois, Karl Weierstrass, and Bernhard Riemann. It was Abel who revolutionized the subject of elliptic functions and opened the flood-gate of the nineteenth-century complex analysis in 1827 with a simple and beautiful remark: "I propose to consider the inverse function." Instead of directly investigating the elliptic integral of the form

$$y = \int^x \frac{dt}{\sqrt{(1-m^2t^2)(1+n^2t^2)}}, \tag{6.3.54}$$

he used this integral as the definition of a new function $y = f(x)$ and then examined the properties of this function by studying the corresponding inverse function $x = f^{-1}(y) = F(y)$. Abel's first major discovery was the double periodicity of the inverse function which became known as the *elliptic functions*. The more general elliptic integral is of the form

$$w(x) = \int^x R(x, \sqrt{P(x)})dx, \tag{6.3.55}$$

where $y^2 = P(x)$ is a polynomial of third or fourth degree with distinct roots and $R(x, y)$ is a rational function of x and y.

When $P(x)$ is of degree $n > 5$, integral (6.3.55) is called the *hyperelliptic integral*. Replacing x by a complex number z in (6.3.55), the hyperelliptic integral is a function $w(z)$ of a complex variable z. Abel investigated the inverse function z of w and could not solve the problem completely because w is often a multiple-valued function of z. However, Jacobi investigated the particular hyperelliptic integrals

$$w = \int_0^z \frac{dz}{\sqrt{P(z)}} \quad \text{and} \quad w = \int_0^z \frac{z\,dz}{\sqrt{P(z)}}, \tag{6.3.56}$$

where $P(z)$ is a polynomial in z of degree $n > 5$. Like Abel, Jacobi was not able to determine the single-valued inverse function z of w. Subsequently, Galois began to study of a generalization of the elliptic and hyperelliptic integrals, but the more significant first step was initiated by Abel in his 1826 paper on more general elliptic integral of the form

$$w = \int^{z} R(u, z) dz, \tag{6.3.57}$$

where u and z are connected by a general algebraic equation $F(u, z) = 0$ instead of $u^2 = P(z)$ which is a polynomial in z of degree $n > 5$. The integral (6.3.57) is called an *Abelian integral* which is a generalization of elliptic and hyperelliptic integrals. Although Abel could not make an extensive study of Abelian integrals, he gave a non-rigorous proof of a fundamental theorem in his Paris paper in 1826 and published its statement in 1829 *Crelle's Journal*. Abel's key theorem is a key broad generalization of the addition theorem for elliptic integrals. He also demonstrated that the sum of the integrals of the form (6.3.57) can be expressed in terms of p such integrals plus algebraic and logarithmic terms, where the number p depends only on the equation $F(x, y) = 0$ and is, indeed, the genus of this equation. He also computed the number p for a few special cases of the general equation $F(x, y) = 0$. He may be considered the founder of the subject of Abelian integrals and Abelian functions. However, he could not identify the full significance of his analysis and results, but he loosely recognized the idea of genus before Bernhard Riemann who was a student of Friedrich Gauss.

It was Riemann and Weierstrass who made significant contributions to establish rigorous theory of Abelian integrals and Abelian functions. Their successors in this field introduced transcendental functions from algebraic functions. Riemann collected together all ideas and results of Jacobi and Abel, which stemmed mainly from real functions, and Weierstrass's new approach to the theory of complex functions. Based on complex function theory, he made a new and remarkable study of the inversion of Abelian integrals (6.3.57). His analysis revealed that the inverse function z of w is not only multi-valued, but often cannot be defined. As in the case of hyperelliptic integrals, Riemann considered sums of p Abelian integrals, and introduced new *Abelian functions* of p variables which are singled-valued and $2p$-tuply periodic. He showed that the Abelian functions are new generalizations of the elliptic functions as they can be expressed in terms of theta functions in p-variables. His investigation of Abelian integrals led to results on what kinds of functions can exist on a Riemann surface determined by an equation $F(u, z) = 0$. The profound work of Riemann was completed by Gustav Roch (1839-1866) in 1864 which led to the celebrated result on functions on a Riemann surface of genus p, known as the *Riemann–Roch theorem*. More precisely, this theorem determines the number of linearly independent meromorphic functions on the surface that have at most a prescribed finite set of poles. On the other hand, Weierstrass was familiar with Abel's work on Abelian integrals from his published papers in 1830s, and learned Jacobi's contributions to elliptic functions from his teacher Gudermann. He also developed a totally new theory of analytic functions of a complex variable based on the method of power series and the process of analytic continuation which he also learned from Gudermann in 1840s. Subsequently, he also made an extensive study of Abelian integrals and functions in 1860s.

Chapter 7

Euler's Contributions to the Infinite Series and the Zeta Function

> "No mathematician ever attained such a position of undisputed leadership in all branches of mathematics, pure and applied, as Euler did for the best part of the eighteenth century."

André Weil

> "The construction and acceptance of the theory of divergent series is another striking example of the way in which mathematics has grown. It shows, first of all, that when a concept or technique proves to be useful even though the logic of it is confused or even nonexistent, persistent research will uncover a logical justification, which is truly an afterthought. It also demonstrates how far mathematicians have come to recognize that mathematics is man-made. The definitions of summability are not the natural notion of continually adding more and more terms, the notion which Cauchy merely rigorized; they are artificial. But they serve mathematical purposes, including even the mathematical solution of physical problems; and these are now sufficient grounds for admitting them into the domain of legitimate mathematics."

Morris Kline

7.1 Introduction

Before Euler, many great mathematicians including John Wallis, Isaac Newton, Brook Taylor, Gottfried Leibniz, Colin Maclaurin, James Gregory and James Stirling (1692-1770) made brilliant contributions to the subject of infinite series and infinite products. They had demonstrated the series representations of the constants π and e and the use of infinite series and infinite products to represent functions. In 1665, Wallis obtained the infinite product representation of π as

$$\frac{\pi}{2} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdots = \prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)}. \tag{7.1.1}$$

In 1674, at age of 28, Leibniz used the quadrature of a circle to discover perhaps the most remarkable slowly convergent infinite series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots . \tag{7.1.2}$$

Or, equivalently,

$$\frac{\pi}{4} = \frac{2}{1\cdot 3} + \frac{2}{5\cdot 7} + \frac{2}{9\cdot 11} + \cdots . \tag{7.1.3}$$

Using the quadrature of a hyperbola, he obtained another remarkable series

$$\frac{1}{4}\log 2 = \frac{1}{2\cdot 4} + \frac{1}{6\cdot 8} + \frac{1}{10\cdot 12} + \cdots . \tag{7.1.4}$$

These series led him to believe a deeper connection between the two basic transcendental problems, the quadrature of the circle and the quadrature of the hyperbola.

However, a few years before, James Gregory discovered the infinite series representation of the function $f(x) = \tan^{-1} x$ in the form

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots . \tag{7.1.5}$$

When $x = 1$, this series reduces to the Leibniz series (7.1.2).

On the other hand, in 1664, Newton developed a method of dealing with infinite series, and in 1665, he discovered the general binomial series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots = \sum_{m=0}^{\infty}\frac{n!}{(n-m)!m!}\,x^m, \tag{7.1.6}$$

and the infinite series for $\sin^{-1} x$ as

$$\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots \tag{7.1.7}$$

$$(\sin^{-1} x)^2 = x^2 + \frac{2}{3}\cdot\frac{x^4}{2} + \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{x^6}{3} + \cdots . \tag{7.1.8}$$

The series (7.1.7) and (7.1.8) are discovered by Newton; the rigorous method of proof of them was given by Cauchy.

When $x = \frac{1}{2}$, the series (7.1.7) reduces to the Newton numerical series

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 8} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 32} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 128} + \cdots . \tag{7.1.9}$$

Using the first quadrant of a circular arc with center at the origin and radius unity, Newton derived the relation $y = \sqrt{1 - x^2}$ and

$$dz = \frac{dx}{y} = \frac{dx}{\sqrt{1 - x^2}} = \left[1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots\right]. \tag{7.1.10}$$

Integrating (7.1.10) from 0 to x led him to derive the series (7.1.7) for $z = \sin^{-1} x$. Using the Newton method of inversion described by Dunham in his book (2005, pages 17-18) the series (7.1.7) for $z = \sin^{-1} x$ can be transformed into the series for $x = \sin z$ as

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots . \tag{7.1.11}$$

The work of Gregory and Newton led to the representation of a function $f(x)$ in terms of finite differences which is known as the *Gregory–Newton series* in the form

$$f(x + h) = f(x) + \frac{h}{c} \triangle f(x) + \frac{\frac{h}{c}\left(\frac{h}{c} - 1\right)}{2!} \triangle^2 f(x) + \cdots \tag{7.1.12}$$

where $\triangle f(x) = f(x+c) - f(x)$, $\triangle^2 f(x) = \triangle\left[\triangle f(x)\right] = \triangle f(x+c) - \triangle f(x)$, $\triangle^3 f(x) = \triangle\left[\triangle^2 f(x)\right] = \triangle^2 f(x+c) - \triangle^2 f(x)$, $\cdots$. The series (7.1.12) is the first major result in the calculus of finite differences.

Brook Taylor used the Gregory–Newton series (7.1.12) to develop the most remarkable method for expansion of a function into an infinite series

$$f(x + h) = f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots . \tag{7.1.13}$$

This is celebrated *Taylor series expansion* of a function $f(x)$ in 1715.

James Stirling also obtained Maclaurin's power series representation of algebraic function in 1717 and of general function in his *Methodus Differentialis* of 1730. Incidently, it may be pointed out that Newton was not the first to discover the series for $\sin x$. In 1545, the Indian mathematician Nilakantha (1445-1545) described the series for $\sin x$ and gave credit for the discovery of series for $\sin x$ to his more remote predecessor Madhava.

It is evident from the above introduction that almost all of the great mathematicians of the eighteenth century made remarkable contributions to the subject of infinite series without any special attention to the question of convergence and divergence. Indeed, Euler earned the crowning glory from his work on infinite series and infinite products. His extensive research on this subject began from his solution of the Basel problem and the discovery of the general infinite series for the zeta function. This problem dealt with the determination of the sum of the squares of the reciprocals of integers. In 1730s, Euler *first* solved the Basel problem in four different ways which was his first really major mathematical accomplishment. The sum of the Basel series was one of the first examples of an infinite series and represented the value of the Euler zeta function $\zeta(s)$ at $s = 2$. So, the solution of the Basel problem clearly motivated him to discover the zeta function $\zeta(s)$ defined by an infinite series for real s. So, the discovery of the zeta function was another of Euler's ingenious work with a beauty comparable to that of great art or music.

7.2 Euler and the Infinite Series

Historically, the celebrated Basel problem dealt with the determination of the exact sum of the squares of the reciprocals of the integers

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}. \tag{7.2.1}$$

This sum was the value of the Euler zeta function $\zeta(s)$ at $s = 2$. In around 1730, Euler generalized (7.2.1) by first introducing the zeta function defined by the convergent infinite series for real $s > 1$ in the form

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots \infty. \tag{7.2.2}$$

When $s = 1$, the series (7.2.2) gives the celebrated harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots . \tag{7.2.3}$$

This series diverges to infinity very slowly. Euler's research on the harmonic series strongly motivated him to discover a new mathematical constant, now known as the *Euler universal constant*, γ. In order to define γ, he started with an infinite series

$$\log\left(1 + \frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \cdots . \tag{7.2.4}$$

Or, equivalently,

$$\frac{1}{x} = \log\left(\frac{x+1}{x}\right) + \frac{1}{2x^2} - \frac{1}{3x^3} + \frac{1}{4x^4} - \cdots . \tag{7.2.5}$$

Substituting for $x = 1, 2, 3, \cdots, n$ in (7.2.5) gives n numerical series which are added together after cancelling log terms to obtain

$$\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) = \log(n+1) + \frac{1}{2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right)$$
$$-\frac{1}{3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\cdots+\frac{1}{n^3}\right)+\frac{1}{4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\frac{1}{n^4}\right)-\cdots.$$

Euler wrote this result in the form

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} = \log(n+1) + C, \tag{7.2.6}$$

where C denotes the sum of the infinite set of finite arithmetic series. He then calculated the value of C for large n and obtained $C \sim 0.577218\cdots$. This constant C is now known as the *Euler universal constant*, and is denoted by γ. In order to obtain a more precise representation of γ, we subtract $\log n$ from sides of (7.2.6) and note that $\log(n+1) - \log n = \log\left(1+\frac{1}{n}\right)$ which tends to zero as $n \to \infty$ so that the Euler constant γ can be written as

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}-\log n\right). \tag{7.2.7}$$

However, it has not yet been proved whether γ is rational or irrotational.

In one of his paper in 1739, Euler obtained a series infinite both directions as

$$\cdots+\frac{1}{x^2}+\frac{1}{x}+1+x+x^2+x^3+\cdots = 0. \tag{7.2.8}$$

This follows from combining the results

$$\frac{x}{1-x} = x(1-x)^{-1} = x + x^2 + x^3 + \cdots \tag{7.2.9}$$

and

$$\frac{x}{x-1} = \left(1-\frac{1}{x}\right)^{-1} = 1+\frac{1}{x}+\frac{1}{x^2}+\cdots. \tag{7.2.10}$$

The series (7.2.9) is valid for $x < 1$ and series (7.2.10) is valid for $x > 1$. There is no value of x for which they are both correct.

Using the Gregory series (7.1.5) and the trigonometric identity

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}, \tag{7.2.11}$$

Euler derived a new *Euler series* in the form

$$\frac{\pi}{4} = \left[\frac{1}{2}-\frac{1}{3}\left(\frac{1}{2}\right)^3+\frac{1}{5}\left(\frac{1}{2}\right)^5-\cdots\right]$$
$$+\left[\frac{1}{3}-\frac{1}{3}\left(\frac{1}{3}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5-\cdots\right]. \tag{7.2.12}$$

We can also derive another infinite series

$$\tan^{-1} x = \frac{x}{1+x^2}\left[1 + \frac{2}{3}\cdot\frac{x^2}{1+x^2} + \frac{2}{3}\cdot\frac{4}{5}\left(\frac{x^2}{1+x^2}\right)^2 + \cdots\right] \qquad (7.2.13)$$

from the following infinite series

$$\frac{\sin^{-1} y}{\sqrt{1-y^2}} = y + \frac{2}{3}y^3 + \frac{2\cdot 4}{3\cdot 5}y^5 + \cdots. \qquad (7.2.14)$$

Substituting $\sin^{-1} y = \theta$ so that $y = \sin\theta$ in (7.2.14) gives

$$\theta = \sin\theta\cos\theta\left(1 + \frac{2}{3}\sin^2\theta + \frac{2\cdot 4}{3\cdot 5}\sin^4\theta + \cdots\right). \qquad (7.2.15)$$

Putting $\tan^{-1} x = \theta$ so that $\tan\theta = x$, $\sin\theta = \frac{x}{\sqrt{1+x^2}}$ and $\cos\theta = \frac{1}{\sqrt{1+x^2}}$ in (7.2.15) yields (7.2.13). We next use Gregory's series (7.1.5) and identity (7.2.11) to obtain a new series for $\frac{\pi}{4}$:

$$\frac{\pi}{4} = \frac{4}{10}\left[1 + \frac{2}{3}\cdot\frac{2}{10} + \frac{2}{3}\cdot\frac{4}{5}\left(\frac{2}{10}\right)^2 + \cdots\right] + \frac{3}{10}\left[1 + \frac{2}{3}\cdot\frac{1}{10} + \frac{2}{3}\cdot\frac{4}{5}\left(\frac{1}{10}\right)^2 + \cdots\right]. \qquad (7.2.16)$$

Euler also used the identity

$$\frac{\pi}{4} = 5\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{3}{79},$$

to derive another new series for $\frac{\pi}{4}$.

Similarly, using Gregory's series (7.1.15) or the series (7.2.13) and the following identities

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}, \qquad \frac{\pi}{4} = 4\ tan^{-1}\frac{1}{5} - \tan^{-1}\left(\frac{1}{239}\right), \qquad (7.2.17)$$

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}, \qquad (7.2.18)$$

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}, \qquad (7.2.19)$$

we can derive several new infinite series.

It was Euler who first made a serious attempt to study divergent series in a systematic manner. In his letter to Goldback in 1745, Euler asserted that the sum of a power series is the value of the function from which the series is constructed. He also stated that every infinite series must have a sum but since the word sum means the ordinary process of addition, and this process does not give the sum for the case of divergent series such as the Leibniz alternating series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots, \tag{7.2.20}$$

and the harmonic series (7.2.3). Perhaps, Euler preferred the word value for the sum of a divergent series.

He started with the series expansion

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots, \tag{7.2.21}$$

and put $x = 1$ so that the sum becomes $\frac{1}{2}$ of the Leibniz series (7.2.20).

Differentiating (7.2.21) gives the expansion

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \cdots + (-1)^n (n+1) x^n + \cdots, \tag{7.2.22}$$

which, when $x = 1$, Euler found the sum $\frac{1}{4}$ of the series

$$\frac{1}{4} = 1 - 2 + 3 - 4 + \cdots + (-1)^n (n+1) + \cdots, \tag{7.2.23}$$

and so on. Such examples led Euler to formulate his general principle of assigning to any series its sum as the value at $x = 1$ of its generating function. Euler was fascinated most by the divergent series, the one which he referred to as *divergent series par excellence* in his 1760 paper. This is his *factorial* series

$$1 - 1! + 2! - 3! + 4! - 5! + 6! - \cdots + (-1)^n n! + \cdots. \tag{7.2.24}$$

More generally, the power series

$$1 - 1!\,x + 2!\,x^2 - 3!\,x^3 + 4!\,x^4 - \cdots + (-1)^n n!\,x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n n!\,x^n, \tag{7.2.25}$$

cannot be used to study (7.2.24) as the power series (7.2.25) diverges everywhere in the complex plane except at $x = 0$. Euler developed an ingenious method to find its accurate numerical value. He used differential equation method and a method of summation to study the following divergent factorial series

$$f(x) = 1 - 1!\,x + 2!\,x^2 - 3!\,x^3 + \cdots. \tag{7.2.26}$$

In his correspondence with Nicholas Bernoulli, Euler first showed that

$$y(x) = x\,f(x) = x - 1!\,x^2 + 2!\,x^3 - 3!\,x^4 + \cdots, \tag{7.2.27}$$

satisfies the first order differential equation

$$\frac{dy}{dx} + Py = Q, \qquad P = \frac{1}{x^2} \quad \text{and} \quad Q = \frac{1}{x}. \tag{7.2.28}$$

The method of integrating factor $\mu = \exp\left[\int P\,dx\right] = \exp\left(-\frac{1}{x}\right)$ gives the general solution.

$$y(x) = e^{\frac{1}{x}} \int x^{-1} \exp\left(-\frac{1}{x}\right) dx + C \exp\left(\frac{1}{x}\right), \tag{7.2.29}$$

where C is an arbitrary constant. Setting $C = 0$, we obtain

$$y(x) = e^{\frac{1}{x}} \int_0^x \frac{1}{t}\, e^{-\frac{1}{t}}\, dt$$

so that

$$f(x) = \frac{1}{x}\, e^{\frac{1}{x}} \int_0^x \frac{1}{t}\, e^{-\frac{1}{t}}\, dt. \tag{7.2.30}$$

Putting $x = 1$, Euler obtained the sum of the series (7.2.24) numerically as

$$f(1) = \int_0^1 \frac{1}{t} \exp\left(1 - \frac{1}{t}\right) dt \sim 0.59637255. \tag{7.2.31}$$

Similarly, Euler evaluated another divergent series

$$\begin{aligned} g(x) &= x - 1\,x^3 + 1.3\,x^5 - 1.3.5\,x^7 + \cdots \\ &= \exp\left(\frac{1}{2x^2}\right) \int_0^x \exp\left(-\frac{1}{2t^2}\right) t^{-2} dt, \end{aligned} \tag{7.2.32}$$

where $g(x)$ satisfies the first order differential equation $x^3 g' + g = x$ which has the integrating factor $\exp\left(-\frac{1}{2x^2}\right)$.

Thus, he obtained the sum of the divergent series as

$$1 - 1 + 1.3 - 1.3.5 + 1.3.5.7 - \cdots = \int_0^1 \exp\left[\frac{1}{2}\left(1 - t^{-2}\right)\right] t^{-2} dt. \tag{7.2.33}$$

Replacing $n!$ by the gamma function as

$$n! = \Gamma(n+1) = \int_0^\infty e^{-t} t^n dt,$$

Euler treated the divergent series (7.2.26) in the form

$$f(x) = \int_0^\infty e^{-t} dt - x \int_0^\infty t e^{-t} dt + x^2 \int_0^\infty t^2 e^{-t} dt - \cdots. \tag{7.2.34}$$

A formal interchange of summation and integration leads to

$$f(x) = \int_0^\infty e^{-t} \left(1 - xt + x^2 t^2 - \cdots\right) dt = \int_0^\infty \frac{e^{-t}}{1 + xt} dt. \tag{7.2.35}$$

Consequently, the integral solution $y(x) = x\,f(x)$ of the differential equation (7.2.28) is

$$y(x) = x\,f(x) = \int_0^\infty \frac{x\,e^{-t}dt}{1+xt}. \tag{7.2.36}$$

Using his rules for conversion of convergent infinite series into continued fractions, Euler transformed his divergent series (7.2.26) into the continued fraction

$$\frac{x}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{2x}{1+}\,\frac{3x}{1+}\,\frac{3x}{1+}\cdots. \tag{7.2.37}$$

It follows from the above that Euler made two major contributions. First, he obtained an integral as the *sum* of the divergent series (7.2.26) and then proved an asymptotic approximation of the integral. Second, he developed a method of converting divergent series into continued fractions. Then, he utilized the continued fraction for $x = 1$ to find a value of the sum of the divergent numerical series (7.2.24).

On the other hand, Edmond Laguerre (1834-1886), a famous French mathematician, used an argument similar to that of Euler to determine the value of a divergent series

$$f(x) = 1 + x + 2!x^2 + 3!x^3 + \cdots \tag{7.2.38}$$

Laguerre showed that his series (7.2.38) formally satisfies the differential equation

$$x^2\,\frac{d^2f}{dx^2} + (x-1)f(x) = -1 \tag{7.2.39}$$

and so, the function $f(x)$ represents an integral solution

$$f(x) = \int_0^\infty \frac{e^{-t}dt}{1-xt}, \tag{7.2.40}$$

of the equation (7.2.39). He then showed that the integral solution can be converted to the continued fraction like (7.2.37).

The method of differentiation or integration is often used to reduce a given unfamiliar series to a known one. This method may often be employed even if the infinite series does not contain a variable. There are cases as stated above in which they cannot be rigourously justified and may even lead to divergent series. For example

$$f(x) = \frac{1}{1\cdot 2} + \frac{x}{2\cdot 3} + \frac{x^2}{3\cdot 4} + \frac{x^3}{4\cdot 5} + \cdots, \quad f(0) = \frac{1}{2}, \tag{7.2.41}$$

so that

$$x^2 f(x) = \frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \frac{x^5}{4\cdot 5} + \cdots . \tag{7.2.42}$$

Differentiating (7.2.42) twice gives the Cauchy–Euler nonhomogeneous differential equation

$$\left(x^2 f(x)\right)'' = 1 + x + x^2 + x^3 + \cdots = (1-x)^{-1}. \tag{7.2.43}$$

Integrating twice gives the solution of (7.2.43) as

$$f(x) = \frac{1}{x} + \frac{1}{x^2}(1-x)\ln(1-x). \tag{7.2.44}$$

We consider another series

$$f(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{1-x}. \tag{7.2.45}$$

Integrating term by term leads to

$$\int_0^x f(t)dt = x + x^2 + x^3 + \cdots = \frac{x}{1-x}. \tag{7.2.46}$$

Then differentiating (7.2.46) gives the Newton binomial series (7.2.22) with x replaced by $-x$ or (7.2.45)

$$f(x) = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{1}{(1-x)^2}. \tag{7.2.47}$$

To find the sum of the numerical series

$$S = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots , \tag{7.2.48}$$

we consider the power series

$$f(x) = \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{3x^4}{4!} + \cdots , \tag{7.2.49}$$

so that $S = f(1)$. Differentiating (7.2.49) yields

$$f'(x) = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots = xe^x, \tag{7.2.50}$$

and then integrating gives

$$f(x) = \int_0^x x\, e^x dx = x\, e^x - e^x + 1. \tag{7.2.51}$$

Obviously, the sum of the numerical series (7.2.48) is

$$S = f(1) = 1.$$

It follows from the binomial series (7.1.6), that is,

$$f(x) = (1+x)^n = \sum_{m=0}^{\infty} \frac{n!}{(n-m)!}\, x^m \tag{7.2.52}$$

with $x = 1$ gives the sum of the numerical series

$$S = 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots = f(1) = 2^n. \tag{7.2.53}$$

It was Thomas Jan Stieltjes (1856-1894) who made a serious attempt to investigate continued fraction expansions of divergent series and its connection with definite integral. He proved that the continued fraction

$$\frac{1}{a_1 z+}\;\frac{1}{a_2+}\;\frac{1}{a_3 z+}\;\frac{1}{a_4+}\;\frac{1}{a_5 z+}\;\cdots\;\frac{1}{a_{2n}+}\;\frac{1}{a_{2n+1} z+}\;\cdots, \tag{7.2.54}$$

where a_n are positive real numbers and z is complex, converges to a function $F(z)$ provided the series $\sum_{n=1}^{\infty} a_n$ diverges and $F(z)$ is analytic in the complex plane except at $z = 0$ and along the negative real axis so that $F(z)$ can be represented by the Stieltjes integral

$$F(z) = \int_0^{\infty} \frac{d\phi(t)}{t+z}. \tag{7.2.55}$$

It is well known that the continued fraction (7.2.54) formally leads to an infinite series

$$\frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \frac{c_3}{z^4} + \cdots, \tag{7.2.56}$$

where c_n are positive, and conversely, to every series (7.2.56) there corresponds to a continued fraction (7.2.54) with positive a_n. Stieltjes demonstrated how to determine c_n from a_n, and in the case where $\sum_{n=1}^{\infty} a_n$ is divergent, the ratio c_n/c_{n-1} increases and the series (7.2.56) diverges for all z.

If $\phi(t)$ is differentiable, then (7.2.55) reduces to the form

$$F(z) = \int_0^{\infty} \frac{f(t)dt}{t+z}, \tag{7.2.57}$$

where $d\phi(t) = f(t)dt$. To every divergent series (7.2.56) where $\sum_{n=1}^{\infty} a_n$ is divergent it corresponds to an integral of the form (7.2.57). Given the series (7.2.56), finding $f(t)$ is called the *Stieltjes problem*. A formal expansion of the integral determines the constants

$$c_n = \int_0^{\infty} t^n f(t)dt, \quad n = 0, 1, 2, \cdots. \tag{7.2.58}$$

Thus, given c_n, it is possible to determine $f(t)$ from the infinite set of equations (7.2.58). This is known as the famous *Stieltjes moment problem.* However, this problem does not admit a unique solution as Stieltjes himself constructed a function $f(t) = \exp\left(-t^{\frac{1}{4}}\right)\sin t^{\frac{1}{4}}$ which makes $c_n = 0$ for all values of n. So, additional conditions are needed to find a unique solution of the problem.

Emile Borel (1871-1956), one of the leading French mathematicians of the nineteenth century, provided the systematic development of the theory of divergent series with a generalization of Ernesto Cesáro's (1859-1906) summability definition of divergent series. If the above argument of Laguerre is applied to the power series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \tag{7.2.59}$$

with a finite radius of convergent including zero, then the integral

$$\int_0^{\infty} e^{-t} f(xt)dt = S, \tag{7.2.60}$$

is called the *Borel sum* of (7.2.59) provided the integral (7.2.60) exists, where

$$f(xt) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(xt)^n. \tag{7.2.61}$$

Borel used integral (7.2.60) and the associated series (7.2.61) to develop his theory of divergent series. He also introduced the idea of absolute summability, and proved that the absolutely summable series can be treated as a convergent series. In spite of serious objection raised by A. L. Cauchy and N. H. Abel, the concept of summability gained some acceptance because divergent series and integrals arise very often in problems in mathematical sciences. There have been several summability methods developed to deal with divergent series and integrals. Several great mathematicians including Ernesto Cesáro, Leopold Fejér (1880-1959), Niels Henrik Abel, Otto Hölder (1859-1937), Friedrich Riesz (1880-1956), Karl Weierstrass, G. H. Hardy, John Littlewood and Norbert Wiener developed different types of summability methods for series and integrals to generalize the idea of convergence. However, the modern theory of divergent series began in 1880 with the famous paper of Georg Frobenius (1849-1917). During the nineteenth and twentieth centuries, considerable progress has been made on divergent series and integrals in order to obtain mathematical solution of physical problems.

It is true that there was considerable controversy about the sum of divergent series in the seventeenth century because appropriate foundation of analysis has not been laid out at that time. It is very pertinent or appropriate to include the comments on divergent series by a renowned British pure mathematician, G. H. Hardy in his famous book on *Divergent Series* published in 1948 as follows:

"... it does not occur to a modern mathematician that a collection of mathematical symbols should have a 'meaning' until one has been assigned to it by definition. It was not a triviality even to the greatest mathematicians of the eighteenth century. They had not the habit of definition: it was not natural to them to say that, in so many words, 'by X we *mean* Y' ... but it is broadly true to say that mathematicians before Cauchy asked not 'How shall we *define* $1-1+1-\cdots$' but 'What *is* $1-1+1-\cdots$?', and that this habit of mind led them into unnecessary perplexities and controversies which were often really verbal."

While discussing Euler's remarks in his interesting letter to Goldback in 1745, G. H. Hardy also said:

"It is a mistake to think Euler as a 'loose' mathematician, though his language may sometimes seem loose to modern ears; and even his language sometimes suggests a point of view far in advance of the general ideas of his time. ... Here, as elsewhere, Euler was substantially right. The puzzles of the time about divergent series arose mostly, not from any particular mystery in divergent series as such, but from disinclination to give formal definitions and from the inadequacy of the current theory of functions."

Clearly, Euler was very much ahead of his time with regard to his work on divergent series. In spite of Euler's great success in finding the value or sum of a divergent series, Euler has been criticized for his lack of mathematical rigor when he dealt with divergent infinite series. However, the era of Leibniz and Euler was more dominated by profound intuition rather than more logical and rigorous reasoning of modern mathematics. In spite of serious objection of Cauchy, Abel and S. D. Poisson (1781-1840) successfully used divergent series and integrals to represent the free surface elevation function in the theory of waves produced in deep water by a local disturbance acting on the free surface of water. On the other hand, Abel introduced a method of summability to generalize the notion of convergence more than a century later after Euler's work on divergent series. Abel's summability is stronger than the Cesáro method of summation. If a series is Cesáro summable, it is always Abel summable to the same sum. However, the series

$$1 - 2 + 3 - 4 + 5 - \cdots = \sum_{n=0}^{\infty} (-1)^n (n+1), \tag{7.2.62}$$

is Abel summable to $\frac{1}{4}$, since

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = \frac{1}{(1+x)^2}. \tag{7.2.63}$$

But the series is not Cesáro summable.

Finally, it may not be out of place to mention the asymptotic expansion of the Euler integral as follows:

$$\int_x^{\infty} t^{-1} e^{x-t} dt \sim \frac{1}{x} - \frac{1}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots \text{ as } \quad x \to \infty. \tag{7.2.64}$$

In spite of early criticisms and controversies, divergent series and integrals have been found to be useful in the representation and asymptotic approximation of functions. From mathematical and physical points of view, Euler's ingenious work was very useful and has served as the foundation of more modern theory of divergent series and integrals with applications.

7.3 Euler's Zeta Function

After solving the famous Basel problem in 1735, Euler introduced the zeta function, $\zeta(s)$ by the infinite series (7.2.1) in around 1737. He then continued his research for finding the value of $\zeta(2n)$ for the natural number $n \geq 1$. Almost 110 years before Riemann's discovery of $\zeta(s)$ for complex $s = z = x + iy$ in 1859, Euler used the summation of divergent series and mathematical induction to discover a remarkable functional equation for the zeta function in 1749 in the form

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s), \tag{7.3.1}$$

where $\Gamma(s)$ is the Euler gamma function. Or, equivalently, the functional equation (7.3.1) takes the form $\Lambda(s) = \Lambda(1-s)$, where $\Lambda(s)$ is equal to the left hand side of (7.3.1).

It follows from the definition (7.2.2) of $\zeta(s)$ that

$$\begin{aligned}\left(1 - \frac{1}{2^s}\right) \zeta(s) &= \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) - \left(\frac{1}{2^s} + \frac{1}{4^s} + \cdots\right), \\ &= \frac{1}{1^s} + \frac{1}{3^s} + \frac{1}{5^s} + \cdots \infty.\end{aligned}$$

This means that the multiplication of $\zeta(s)$ by $(1 - 2^{-s})$ results in the elimination of all the terms in which n is a multiple of 2 from the original series for $\zeta(s)$. Similarly,

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\zeta(s) = \left(1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots\right) - \left(\frac{1}{3^s} + \frac{1}{9^s} + \cdots\right),$$

where all the terms in which n is a multiple of 2 or 3 have been eliminated from the original series for the zeta function. Proceeding inductively, it turns out that

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\cdots\left(1 - \frac{1}{p^s}\right)\zeta(s) = 1 + \sum{}' \frac{1}{n^s}, \qquad (7.3.2)$$

where $\sum{}'$ begins with the first prime after p, and

$$\left|\sum{}' \frac{1}{n^s}\right| < \sum{}' \frac{1}{n^{1+\delta}} < \sum_{n=p+1}^{\infty} \frac{1}{n^{1+\delta}} \to 0, \qquad (7.3.3)$$

as $p \to \infty$ for any $\delta > 0$.

Consequently, for $Re\, s > 1$,

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \qquad (7.3.4)$$

where the product is taken for all primes p.

Or, equivalently,

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right). \qquad (7.3.5)$$

Using his neat analytic proof given above, Euler established his celebrated identity (7.3.4) or (7.3.5) which expresses the zeta function as an infinite product extended over prime numbers only. The Euler identity appeared for the first time in Euler's famous treatise *Introductio in Analysin Infinitorum* published in 1748. This is a remarkable discovery of Euler which was the starting point of the *Riemann Hypothesis*. Evidently, the zeta function is closely associated with the distribution of prime numbers and plays a fundamental role in number theory and analysis.

One of the remarkable consequences of the identity (7.3.4) is that it is an analytic expression of the fundamental theorem of arithmetic. Indeed, each factor of the product $(1 - p^{-s})^{-1}$ can be expressed as a convergent geometric series

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots + \frac{1}{p^{Ns}} + \cdots.$$

Thus, the product

$$\prod_{p_k}\left(1+\frac{1}{p_k^s}+\frac{1}{p_k^{2s}}+\cdots+\frac{1}{p_k^{Ns}}+\cdots\right)$$

can be written in increasing order $p_1 < p_2 < \cdots$. We formally calculate the product as a sum of terms, each term generating from a term p_j^{-ks} in the sum corresponding to p_j with a particular k which depends on j, and with $k = 0$ for j sufficiently large. Consequently, the resulting product is

$$\frac{1}{\left(p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}\right)^s}=\frac{1}{n^s},$$

where the integer $n(\geq 1)$ is written uniquely as a product of primes $n = p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m}$. Thus, the product is equal to

$$\sum_{n=1}^{\infty}\frac{1}{n^s}=\zeta(s).$$

If n is a positive integer and a_k^s are positive integers, then $n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$, and

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}, \tag{7.3.6}$$

where $\mu(n)$ is the *Möbius function* defined by

$$\mu(n)=\begin{cases}1, & n=1\\ (-1)^k, & \text{if } a_1=a_2=\cdots a_k=1\\ 0, & \text{otherwise}\end{cases}. \tag{7.3.7}$$

Furthermore, if $d(n)$ denotes the number of divisors of n (including 1 and n) and $\sigma_k(n)$ denotes the sum of the kth powers of the divisors of n, that is, $\sigma_k(n) = \sum_{d|n} d^k$, then

$$\zeta(s)\zeta(s-k)=\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^s},\quad Re\, s>\max(1,1+k), \tag{7.3.8}$$

where $d\,|n$ means d divides n.

Euler's discovery of the product formula (7.3.4) in 1744 was concerned with an unexpected and deep connection between analysis and number theory. Indeed, he proved that the divergence of the harmonic series (7.2.3) implies that the number of primes is infinite and vice versa. In Euler's notation, for $s > 1$, it follows from (7.3.4) that (7.2.2) can be written as

$$\zeta(s)=1+\frac{1}{2^s}+\frac{1}{3^s}+\cdots=\frac{2^s}{2^s-1}\cdot\frac{3^s}{3^s-1}\cdot\frac{5^s}{5^s-1}\cdots\cdot. \tag{7.3.9}$$

This remarkable identity represents the unique factorization property of natural numbers. When $s = 1$, it follows from (7.3.4) that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p} \left(1 - \frac{1}{p}\right)^{-1}. \tag{7.3.10}$$

Or, equivalently,

$$\prod_{p} \left(1 - \frac{1}{p}\right) = 0, \tag{7.3.11}$$

where the product is taken over all primes. Based on his imprecise argument, Euler proved that

$$\sum_{p} \frac{1}{p} = \infty, \tag{7.3.12}$$

where the sum is taken over all primes p, and arrived at the correct conclusion that the number of primes is infinite. Of course, if there were finite number of primes, the above series (7.3.12) would converge automatically. The study of prime numbers had always been his central research topic in number theory with the fundamental focus whether there are infinitely many primes or not. The *Euclid's Elements* contained the solution of the problem in a simple and elegant manner. However, Euler's easy proof of the product formula (7.3.4) laid the foundation of the analytic number theory and stimulated considerable research on the prime number theory and analysis in the nineteenth and twentieth centuries.

As a natural generalization of his great work that there are infinitely many primes, Euler conjectured that any arithmetic progression of the form

$$a, a + h, a + 2h, \cdots, a + nh, \cdots, \tag{7.3.13}$$

where a and h are relatively prime, contained infinitely many primes. It remained an unsolved problem for almost hundred years. Euler's product formula and the conjecture inspired Peter Gustav Lejeune Dirichlet (1805-1859), a student of Friedrich Gauss and then successor to Gauss at Göttingen, to formulate the general problem of primes in arithmetic progression and to generalize the Euler product formula. Using Euler's remarkable insight and imagination, Dirichlet proved that there are infinitely many primes in the arithmetic progressions

$$1, 5, 9, 13, \cdots, (4k+1), \cdots \quad \text{and} \quad 3, 7, 11, 15, \cdots, (4k-1), \cdots. \tag{7.3.14}$$

More remarkable was the Dirichlet's generalization of the Euler zeta function by first introducing the *Dirichlet L-function* defined as

$$L\left(s,\chi\right)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{s}}\quad\text{for}\quad s>1, \tag{7.3.15}$$

where $\chi(n)$ is called the *Dirichlet character* defined by

$$\chi(n)=\left\{\begin{array}{ll}0, & \text{for even } n\\ 1, & \text{for } n=4k+1\\ -1, & \text{for } n=4k+3\end{array}\right\}. \tag{7.3.16}$$

Clearly, $\chi(n)$ is a multiplicative function, that is, $\chi(n)\chi(m)=\chi(mn)$ for all m, $n\in\mathbb{Z}$. In particular, the function $L(s)$ is defined by

$$L(s)=1-\frac{1}{3^{s}}+\frac{1}{5^{s}}-\frac{1}{7^{s}}+\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{s}}, \tag{7.3.17}$$

so that

$$L(1)=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}. \tag{7.3.18}$$

Historically, series (7.3.18) is probably the most simplest representation for $\frac{\pi}{4}$ published in 1670 by a famous British mathematician, James Gregory. The sum of the Gregory series was also rediscovered by the 28-year old Leibniz in 1674 using geometric arguments. The sum of the Gregory series can easily be calculated as the limit of a definite integral

$$\lim_{x\to 1}\int_{0}^{x}\left(1+t^{2}\right)^{-1}dt=\lim_{x\to 1}\left(x-\frac{x^{3}}{3}+\frac{x^{5}}{5}-\cdots\right),\ 0<x<1. \tag{7.3.19}$$

Or, equivalently, in the limit as $x\to 1$,

$$\left[\tan^{-1}t\right]_{0}^{1}=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\cdots\infty=L(1,\lambda). \tag{7.3.20}$$

Since the Dirichlet character function $\chi(n)$ is multiplicative, Dirichlet generalized the Euler product formula in 1837 in the form

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^{s}}=\prod_{p}\left(1-\frac{\chi(p)}{p^{s}}\right)^{-1}\quad\text{for}\quad s>1, \tag{7.3.21}$$

where the product is over all prime numbers p. Based on Euler's ingenious work, Dirichlet product theorem is another remarkable discovery.

Taking logarithm of both sides of (7.3.21) yields

$$\log L(s,\chi)\approx\sum_{p}\frac{\chi(p)}{p^{s}}+O(1). \tag{7.3.22}$$

In the limit as $s \to 1$ with the fact that $L(1,\lambda) = \frac{\pi}{4} \neq 0$ shows that $\sum_p \chi(p)/p^s$ remains bounded. In view of the result that $\sum_p \frac{1}{p}$ diverges, it turns out that there are infinitely many primes of the form $(4k+1)$.

Under certain conditions, the Dirichlet series for real or complex s

$$D(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \tag{7.3.23}$$

can be expressed in terms of a product of *Euler's factors* $E_p(s)$ as follows:

$$D(s) = \prod_p E_p(s), \tag{7.3.24}$$

where the Euler factors are given by

$$E_p(s) = 1 + f(p)\,p^{-s} + f\left(p^2\right)p^{-2s} + f\left(p^3\right)p^{-3s} + \cdots . \tag{7.3.25}$$

When $f(n) = 1$, the Dirichlet function $D(s)$ coincides with the Euler zeta function. It also follows from (7.3.4) that

$$\begin{aligned} \frac{\zeta(2s)}{\zeta(s)} &= \prod_p \left(\frac{1-p^{-s}}{1-p^{-2s}}\right) = \prod_p \left(1+\frac{1}{p^s}\right)^{-1} \\ &= \prod_p \left(1 - p^{-s} + p^{-2s} - p^{-3s} + \cdots\right) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \end{aligned} \tag{7.3.26}$$

where $\lambda(n) = (-1)^\rho$, $\rho = \sum_{m=1}^{k} \alpha_m$ if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorization of n.

In 1749, Euler presented a paper at the Berlin Academy of Sciences and reported a new function related to the zeta function defined by

$$\phi(s) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^s}. \tag{7.3.27}$$

He preferred to work with this alternating series for the phi function rather than the zeta function for better convergence and more accurate numerical calculation. He also discovered the following relation between $\phi(s)$ and $\zeta(s)$, and the famous functional equation for $\phi(s)$:

$$\phi(s) = \left(1 - 2^{1-s}\right)\zeta(s) \tag{7.3.28}$$

$$\pi^s\left(2^{s-1} - 1\right)\phi(1-s) + \left(2^s - 1\right)\cos\left(\frac{\pi s}{2}\right)\Gamma(s)\phi(s) = 0. \tag{7.3.29}$$

This series (7.3.27) for the phi function is convergent for $s > 0$ by the Leibniz test for convergence of alternating series. But, if $s \leq 0$, the series diverges and so, Euler's computation reveals the definition

$$1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots = \lim_{x \to 1-} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n^s}, \qquad s \leq 0. \tag{7.3.30}$$

If $s = m$ is a positive integer, Euler showed that the power series (7.3.30) represents a rational function with $(1+x)^{m+1}$ as its denominator and so its limit as $x \to 1$ exists and can be explicitly calculated.

Euler was not successful to verify (7.3.28) for all s. For $s = 1$, it follows from (7.3.27) that

$$\phi(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2. \tag{7.3.31}$$

The identities (7.3.28) and (7.3.29) can be combined to obtain another famous functional equation for real s in the form

$$\zeta(1-s) = \pi^{-s} 2^{1-s} \Gamma(s) \zeta(s) \cos\left(\frac{\pi s}{2}\right). \tag{7.3.32}$$

One hundred years later, Riemann gave a rigorous proof of this equation in 1859 for complex $s = x + iy$.

Without calculators or computers, in 1731 Euler developed a new method for computation of $\zeta(2)$ based on the infinite series in the form

$$\frac{1}{x} \log(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^{n-1}. \tag{7.3.33}$$

Integrating with respect to x form 0 to 1 gives

$$\zeta(2) = -\int_0^1 \frac{1}{x} \log(1-x) dx.$$

Putting $1 - x = u$ and writing the integral as the sum of two integrals gives

$$\zeta(2) = -\left[\int_0^x \frac{\log u}{1-u} du + \int_x^1 \frac{\log u}{1-u} du\right] \tag{7.3.34}$$

which is, expanding $(1-u)^{-1}$ in power series,

$$\zeta(2) = \log(1-x) \log x + \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \sum_{n=1}^{\infty} \frac{(1-x)^n}{n^2}. \tag{7.3.35}$$

Using $x = \frac{1}{2}$, Euler obtained the formula in 1733 for computing $\zeta(2)$ up to six decimal places

$$\zeta(2) = (\log 2)^2 + \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \cdot \frac{1}{n^2} \tag{7.3.36}$$

$$\sim 0.480453 + 1.164482 = 1.644934. \tag{7.3.37}$$

However, in 1730, Stirling also computed $\zeta(2)$ to nine decimal places. Thus, the problem of numerical computation of $\zeta(n)$ for higher values of n must have motivated Euler to discover his famous Euler–Maclaurin summation formula in 1732. Using this formula, Euler obtained a more accurate approximate value of $\zeta(2)$, $\zeta(3)$, and $\zeta(4)$. His numerical value for $\zeta(2)$ up to twenty decimal places is recorded below for historical interest:

$$\zeta(2) = 1.64493406684822643647. \tag{7.3.38}$$

Evidently, Euler's original method of computation of $\zeta(n)$ for higher values of n opened a new area of research which is known today as *mathematical computation.*

Euler not only proved the formula (7.3.32) for all integral values of s and verified it for $s = \frac{1}{2}$ and $s = \frac{3}{2}$. He then conjectured that the formula is true for all values of s. Like Euler, Riemann also recognized that the zeta function played a fundamental role in the distribution of primes in number theory. For an arbitrary real number $x \geq 2$, $\pi(x)$ denotes the number of prime numbers less or equal to x. For example, $\pi(8) = 4$, since 2, 3, 5 and 7 are primes and $\pi(11) = 5$. Originally, Euler believed that prime numbers are distributed totally irregularly. In fact, Legendre in 1785, and then Gauss in 1792 independently investigated the asymptotic distribution of $\pi(x)$ for large x through an intensive study of tables of logarithms. The fundamental *Prime Number Theorem* states that, for large number x, the asymptotic formula for $\pi(x)$ is

$$\pi(x) \sim \left(\frac{x}{\ln x}\right) = li\, x \quad \text{as} \quad x \to \infty. \tag{7.3.39}$$

In 1798, Legendre proved the following limiting result

$$\lim_{x\to\infty} \frac{\pi(x)}{x} = 0. \tag{7.3.40}$$

This implies that there are considerably fewer prime numbers than natural numbers. The more striking fact is that the asymptotic distribution of prime numbers is closely associated with the singularity of the zeta function. Using the product formula (7.3.5), it can be shown that for large x, the asymptotic distribution of prime numbers is given by

$$\pi(x) \sim \int_2^x \frac{dt}{\ln(t)} = Li(x). \tag{7.3.41}$$

Although (7.3.39) and (7.3.41) are equivalent, the logarithmic integral (7.3.41) provides a more accurate numerical approximation to $\pi(x)$ than does $(x/\ln x)$. In 1791, Gauss first conjectured result (7.3.41) which was finally proved independently by Jacques Hadamard and Charles de la Valleé–Poussin in 1896. However, in 1914, a great British mathematician, J. E. Littlewood (1885-1977) proved that the difference $Li(x) - \pi(x)$ assumes positive and negative values infinitely often. Although the asymptotic result (7.3.41) can numerically be verified for a very large number of cases, it may not be true for all large x. In 1948, Erdös and Selberg discovered an elementary proof of the prime number theorem.

On the other hand, Riemann made much more contributions than simply determining the asymptotic distribution of prime numbers based on the formula (7.3.5). Taking log of (7.3.5), he obtained

$$\log \zeta(s) = \sum_p \frac{1}{p^s} + \frac{1}{2}\sum_p \frac{1}{p^{2s}} + \frac{1}{3}\sum_p \frac{1}{p^{3s}} + \cdots \tag{7.3.42}$$

and then replaced p^{-ns} by

$$s\int_{p^n}^{\infty} x^{-(s+1)} dx$$

so that

$$\frac{1}{s}\log \zeta(s) = \int_1^{\infty} x^{-(s+1)}\, \Pi(x)\, dx, \tag{7.3.43}$$

where

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi\left(x^{1/2}\right) + \frac{1}{3}\pi\left(x^{\frac{1}{3}}\right) + \cdots \infty. \tag{7.3.44}$$

Riemann then expressed $\Pi(x)$ as the Fourier complex inversion integral in the form

$$\Pi(x) = \int_{c-i\infty}^{c+i\infty} \frac{\log \zeta(s)}{s} x^s ds, \qquad c > 1 \tag{7.3.45}$$

and evaluated it by the residues of the singularities of $\log \zeta(s)$ at $s = 1$ and at the zeros of $\zeta(s)$. It then follows from the inversion of (7.3.44) that

$$\pi(x) = \sum_m \frac{(-1)^\mu}{m}\Pi\left(x^{1/m}\right), \tag{7.3.46}$$

where m consists of all natural numbers not divisible by any square other than one, and μ is the prime factors of m. Thus, the distribution of the prime numbers is closely associated with the zeros of $\zeta(s)$.

Like Euler, Riemann also recognized that the zeta function and its zeros played a fundamental role in the distribution and analysis of primes in number theory. The only zeros outside the critical strip defined by the inequality

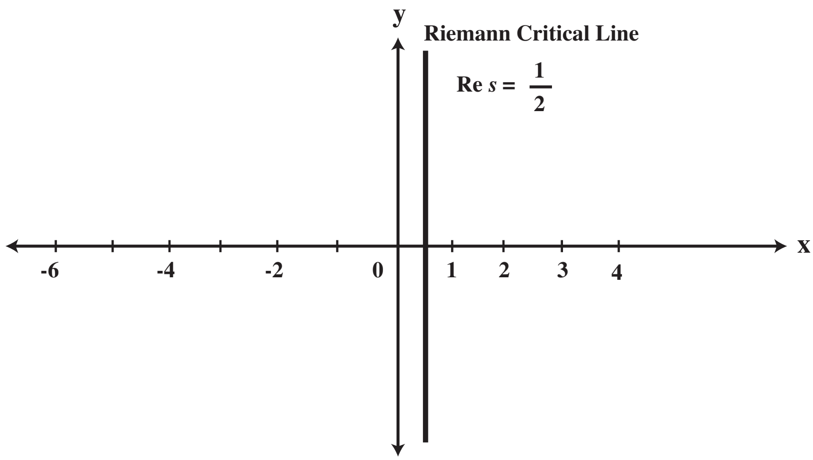


Fig. 7.1 The zeros of the Riemann zeta function.

$0 \leq Re\, s \leq 1$ are at the even negative integers $(s = -2n, \quad n = 1, 2, 3, \cdots)$ known as the *trivial zeros.* He also proved that $\zeta(s)$ is an analytic function in the whole complex s-plane except for a simple pole at $s = 1$ with residue one, and it has no other singularities. In 1859, Riemann formulated his celebrated *Riemann Hypothesis* which states that all non-trivial complex zeros of $\zeta(s)$ lie on the *critical line* $Re\, s = \frac{1}{2}$ in the complex s plane as shown in Figure 7.1. In other words, the complex zeros are at $s = \frac{1}{2} \pm iy$ which are symmetrically located on the critical line. The first few complex zeros are at $y = 14 \cdot 134$, 21, 25, $30 \cdot 5$, 33, $\cdots$. This is the most famous unsolved problem in mathematics. In 1914, G. H. Hardy proved that there are infinitely many zeros of the Riemann zeta function on the *Riemann critical line* $Re\, s = \frac{1}{2}$. Many extensive numerical experiments with supercomputers have given no indication as of yet whether the Riemann hypothesis is true or false. Many recent computations reveal that fifty billion complex zeros lie on the critical line. Furthermore, precise asymptotic estimates show that at least one-third of the zeros of $\zeta(s)$ must lie on the critical line.

The truth of the *Riemann Hypothesis* implies that the deviation of the prime numbers from the asymptotic limit $Li(x)$ is

$$\pi(x) = Li(x) + O\left(\sqrt{x}\ln x\right) \tag{7.3.47}$$

and a single zero off the line $s = \frac{1}{2} + iy$ would change the distribution of primes in a significant way.

In addition to two functions $L(s)$ and $\phi(s)$ defined respectively by (7.3.17) and (7.3.27), Euler preferred to work with another function $\theta(s)$ related to the zeta function defined by

$$\theta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s}, \qquad s > 1. \tag{7.3.48}$$

Obviously, $\theta(s)$ can be expressed in terms of $\zeta(s)$ and $\phi(s)$ in the form

$$\theta(s) = \left(1 - \frac{1}{2^s}\right)\zeta(s), \tag{7.3.49}$$

$$\theta(1-2n) = \frac{(-1)^{n-1}2.(2n-1)!}{\pi^{2n}}\,\phi(2n), \qquad n = 1, 2, 3, \cdots, \tag{7.3.50}$$

where $\theta(m)$, $m = 0, \pm 1, \pm 2, \cdots$ is defined by

$$\theta(m) = \lim_{x \to 1-} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n^m}. \tag{7.3.51}$$

Euler also proved another famous formula for $L(2n+1)$ in the form

$$L(2n+1) = (-1)^n \pi^{2n+1} \frac{E_{2n}}{2^{2n+2}(2n)!}, \tag{7.3.52}$$

where the *Euler numbers*, E_{2n} are defined by the coefficients of the expansion of $\sec x$:

$$\sec x = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{(2n)!} x^{2n}, \tag{7.3.53}$$

so that

$$E_0 = 1, \quad E_2 = 1, \quad E_4 = 5, \quad E_6 = 61, \quad E_8 = 1385, \cdots. \tag{7.3.54}$$

In 1734, Euler discovered another remarkable formula for $\zeta(2n)$ given by

$$\zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots = \frac{(2\pi)^{2n}(B_{2n})}{2(2n)!}, \tag{7.3.55}$$

where B_n are called the *Bernoulli's numbers* representing the coefficients of the series

$$\frac{z}{e^z - 1} = \frac{B_0}{0!} + \frac{B_1}{1!}z + \frac{B_2}{2!}z^2 + \frac{B_4}{4!}z^4 + \cdots = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \tag{7.3.56}$$

where

$$B_0 = 1,\, B_1 = -\frac{1}{2},\, B_2 = \frac{1}{6},\, B_4 = -\frac{1}{30},\, B_6 = \frac{1}{42}, \cdots, \tag{7.3.57}$$

and

$$B_3 = B_5 = \cdots = 0. \tag{7.3.58}$$

In fact, these Bernoulli numbers were first discovered by Euler and they have widely been used in probability theory of James Bernoulli.

In particular, Euler obtained the following remarkable formulas for the zeta functions as

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \text{and} \quad \zeta(8) = \frac{\pi^8}{9450}. \tag{7.3.59}$$

Since $\zeta(2n)$ is always a rational multiple of π^{2n}, it follows that $\zeta(2)$, $\zeta(4)$, $\zeta(6)$, $\cdots$ are all transcendental, and it fact, they are closely related to Bernoulli's numbers. There were many other open problems which dealt with the values of the zeta function at certain special points. For odd $n = 2m+1 (m \geq 1)$, no formula is known for $\zeta(n)$. This is an open unsolved problem. Is $\zeta(2n+1)$ for positive integers n algebraic or transcendental? The answer has not yet been found for a single value of $(2n+1)$. In 1978, Aprey showed that $\zeta(3)$ is irrotational and then, in 2000, he also proved that $\zeta(2n+1)$ is irrotational for infinitely many positive integers n. But, the general problem for all n remains unresolved.

It also follows from the value of $\zeta(2)$ that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}\left(1 - \frac{1}{4}\right) = \frac{\pi^2}{8}. \tag{7.3.60}$$

For all complex z such that $|z| < \pi$, and in particular, for all $\operatorname{Re} z = x$ between $\pm\pi$,

$$\cot z = \frac{1}{z} - \frac{2S_2}{\pi^2} z - \frac{2S_4}{\pi^4} z^3 - \frac{2S_6}{\pi^6} z^5 - \cdots, \tag{7.3.61}$$

where $S_{2n} = \zeta(2n)$. Also, $\cot z$ can be expressed in terms of Bernoulli's numbers as

$$\cot z = \frac{1}{z} - \frac{2^2 B_1}{2!} z - \frac{2^4 B_2}{4!} z^3 - \cdots - \frac{2^{2n} B_n}{(2n)!} z^{2n-1} - \cdots. \tag{7.3.62}$$

Equating the coefficients of (7.3.61) and (7.3.62) gives

$$\frac{2}{\pi^2} S_2 = \frac{2^2 \cdot B_1}{2!}, \quad \frac{2}{\pi^4} S_4 = \frac{2^4}{4!} B_2, \quad \cdots \quad \frac{2}{\pi^{2n}} S_{2n} = \frac{2^{2n}}{(2n)!} B_n. \tag{7.3.63}$$

Using the values of Bernoulli's numbers (7.3.57) and (7.3.58) leads to the result

$$S_{2n} = \zeta(2n) = \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_n. \tag{7.3.64}$$

Substituting the values of $\cot z$ and $\cot 2z$ from (7.3.61) in the identity $\tan z = \cot z - 2\cot 2z$ gives the series expansion

$$\tan z = \frac{2S_2\left(2^2-1\right)}{\pi^2} z + \frac{2S_4\left(2^4-1\right)}{\pi^4} z^3 + \frac{2S_6\left(2^6-1\right)}{\pi^6} z^5 + \cdots, \quad (7.3.65)$$

which holds for $|z| < \frac{\pi}{2}$, and for all real x between $\pm\frac{\pi}{2}$.

On the other hand, substituting for $\cot\frac{1}{2}z$ and $\cot z$ from (7.3.61) in $\operatorname{cosec} z = \cot\frac{1}{2}z - \cot z$, we obtain

$$\operatorname{cosec} z = \frac{1}{z} + (2-1)\frac{S_2}{\pi^2} z + \frac{\left(2^3-1\right)}{2^2}\frac{S_4}{\pi^4} z^3 + \frac{\left(2^5-1\right)}{2^4}\frac{S_6}{\pi^6} z^5 + \cdots, \quad (7.3.66)$$

where $|z| < \pi$.

Similarly, it can be shown that

$$\sec z = \frac{2^2}{\pi} R_1 + \frac{2^4}{\pi^3} R_3 z^2 + \cdots + \frac{2^{2n+2}}{\pi^{2n+1}} R_{2n+1} z^n + \cdots, \quad (7.3.67)$$

where $|z| < \frac{\pi}{2}$ and

$$R_{2n+1} = \frac{1}{1^{2n+1}} - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \cdots. \quad (7.3.68)$$

For complex z $\left(|z| < \frac{\pi}{2}\right)$, the infinite series

$$\operatorname{sech} z = \frac{E_0}{0!} + \frac{E_1}{1!} z + \frac{E_2}{2!} z^2 + \cdots = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n, \quad (7.3.69)$$

converges, where the coefficients $E_n\,(n = 0, 1, 2, \cdots)$ are called the *Euler numbers*, $E_n = 0$ for odd n and

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \cdots. \quad (7.3.70)$$

More generally, the relation between the Euler number and the Bernoulli number is given by

$$E_{2n} = \frac{4^{2n+1}}{(2n+1)} \left(B_n - \frac{1}{4}\right)^{2n+1}, \qquad n = 1, 2, 3, \cdots. \quad (7.3.71)$$

An integral representation of the zeta function is given by

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\,dt}{(e^t - 1)}, \qquad Re\, s > 1. \quad (7.3.72)$$

In 1734-1735, Euler also discovered many additional beautiful numerical series as follows:

$$L(3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}, \tag{7.3.73}$$

$$\theta(4) = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}, \tag{7.3.74}$$

$$L(5) = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \cdots = \frac{5\pi^5}{1536}, \tag{7.3.75}$$

$$\theta(6) = 1 + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots = \frac{\pi^6}{960}, \tag{7.3.76}$$

$$1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots = \frac{\pi}{2\sqrt{2}}, \tag{7.3.77}$$

and

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \cdots = \frac{2\pi}{3\sqrt{3}}. \tag{7.3.78}$$

Euler's phi function (7.3.27) led him to discover another divergent alternating series for odd values of $m > 0$

$$\phi(-m) = 1 - 2^m + 3^m - 4^m + \cdots, \tag{7.3.79}$$

which is related to the power series

$$R_m(x) = 1 - 2^m x + 3^m x^2 - 4^m x^3 + \cdots. \tag{7.3.80}$$

In fact, $R_0(x) = (1+x)^{-1}$ and for integer $m \geq 0$,

$$R_{m+1}(x) = \frac{d}{dx}\left[x\, R_m(x)\right], \tag{7.3.81}$$

so that, for $m > 0$, $R_m(x)$ assumes the form

$$R_m(x) = (1+x)^{-m-1} P_m(x), \tag{7.3.82}$$

where $P_m(x)$ is a polynomial of degree $m-1$ with integral coefficients. In fact, $R_m(x)$ satisfies the relation

$$R_m\left(\frac{1}{x}\right) = (-1)^{m+1} x^2 R_m(x), \tag{7.3.83}$$

and $\phi(-m) = R_m(1) = 0$ for every positive even integer m. However, for odd $m \geq 1$, $\phi(-m)$ is related to $\zeta(m+1)$ which is then related to the Bernoulli numbers so that

$$\phi(-m) = R_m(1) = \left(2^{m+1} - 1\right) \frac{B_{m+1}}{m+1}, \tag{7.3.84}$$

which is valid for all odd numbers $m > 0$ and also for all even $m > 0$, since both sides of (7.3.84) are zero.

Finally, it follows from the Wallis product formula (7.1.1) that

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right)^{-1} \tag{7.3.85}$$

which is, by taking logarithms,

$$\begin{aligned}
\log\frac{\pi}{2} &= -\sum_{n=1}^{\infty}\log\left(1-\frac{1}{4n^2}\right)\\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k\,(4n^2)^k}, \quad \text{since} \quad \log(1-x) = -\sum_{k=1}^{\infty}\frac{x^k}{k}.\\
&= \sum_{k=1}^{\infty}\frac{1}{k.4^k}\sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \sum_{k=1}^{\infty}\frac{\zeta(2k)}{k.4^k}.
\end{aligned}$$

This is a new series representation for $\log\left(\frac{\pi}{2}\right)$.

7.4 Euler and the Fourier Series

Joseph Fourier is also most celebrated for his discovery of the representation of an arbitrary function over an interval in terms of trigonometric functions, universally known as *Fourier series*. However, it is also historically true that the series expansion of cosine and sine functions goes back to Daniel Bernoulli and to Euler who developed the formulas for the Fourier coefficients by integrals. In the eighteenth century, the mathematical study of such series representation of functions originated in problems of mathematical astronomy, conduction of heat, vibrating strings and membranes, due to the fact that functions involved are periodic and phenomena are largely periodic. Euler also presented the theory of *Fourier integrals* in his own work on wave phenomena including water wave problems.

A Fourier series representation of a periodic function $f(x)$ in an interval $-\ell \leq x \leq \ell$ is an expression of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{\ell}\right) + b_n\sin\left(\frac{n\pi x}{\ell}\right)\right], \tag{7.4.1}$$

where the Fourier coefficients a_n and b_n are given by *Euler's formulas* as

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx, \qquad n = 0, 1, 2, 3, \cdots, \tag{7.4.2}$$

and

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx, \qquad n = 1, 2, 3, \cdots. \tag{7.4.3}$$

However, Euler made no mention about the conditions under which these are necessarily the values of the coefficients a_n and b_n.

In particular, when $\ell = \pi$, then the Fourier series takes the simple form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx), \quad -\pi \le x \le \pi, \tag{7.4.4}$$

where the Euler formulas for a_n and b_n are given by

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad n = 1, 2, 3, \cdots, \tag{7.4.5}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \qquad n = 1, 2, 3, \cdots. \tag{7.4.6}$$

Universally considered as the greatest mathematicians since Carl Friedrich Gauss, Bernhard Riemann began to give a rigorous foundation of the theory of Fourier series by considering the first problem of sufficient conditions for the existence of integrals which give the Fourier coefficients a_n and b_n of a function $f(x)$. In the earlier researches on the Fourier series by Dirichlet, he became interested in this series, particularly in its ability to represent both continuous and discontinuous functions. His superior treatment served as the basis for many later investigations on the convergence or summability of Fourier series. He also proved a convergence theorem of Fourier series which states that the Fourier series of a real continuous periodic function f which has only a finite number of relative maximum and minimum converges everywhere to f.

Subsequently, it became clear that it would be simpler to deal with the complex exponential form of the Fourier series representation

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x}{\ell}\right), \tag{7.4.7}$$

where the Fourier coefficients c_n are given by

$$c_n = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\exp\left(-\frac{in\pi x}{\ell}\right)dx, \qquad n = 0, 1, 2, 3, \cdots. \tag{7.4.8}$$

The convergence or divergence of Fourier series has a long and complex history of over three hundred years. The fundamental question is whether the Fourier series generated by a periodic function f converges to f. The answer is certainly not obvious. If $f(x)$ is 2π-periodic continuous function, then the Fourier series (7.4.4) may converge to f for a given x in $-\pi \le x \le \pi$, but *not for all* x in $-\pi \le x \le \pi$. This leads to the questions of *local convergence* or the behavior of f near a given point x, and of *global convergence* or the overall behavior of a function f over the entire interval $-\pi \le x \le \pi$.

There is another problem that deals with the *mean-square convergence* of the Fourier series to $f(x)$ in $(-\pi, \pi)$, that is, if $f(x)$ is integrable on $(-\pi, \pi)$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_n(x)|^2 dx \to 0 \quad \text{as} \quad n \to \infty, \tag{7.4.9}$$

where $s_n(x)$ is the nth partial sum defined by

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), \tag{7.4.10}$$

which is, by (7.4.5) and (7.4.6)

$$\begin{aligned} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 + 2\sum_{k=1}^{n} (\cos kt + \cos kx + \sin kt + \sin kx)\right] f(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 + 2\sum_{k=1}^{n} \cos k(x-t)\right] f(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x-t) f(t)dt = (f * D_n)(x), \end{aligned} \tag{7.4.11}$$

where $(f * g)(x)$ is the convolution of f and g defined by

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t) f(t)dt \tag{7.4.12}$$

and $D_n(\theta)$ is called the *Dirichlet kernel* defined by

$$D_n(\theta) = 1 + 2\sum_{k=1}^{n} \cos k\theta \tag{7.4.13}$$

$$= 1 + \sum_{k=1}^{n} \left(e^{ik\theta} + e^{-ik\theta}\right) = \sum_{k=-n}^{n} e^{ik\theta}. \tag{7.4.14}$$

This is a finite geometric series with the first term $\exp(-in\theta)$, the ratio $\exp(-i\theta)$ and the last term $\exp(in\theta)$, and hence, the sum is given by

$$D_n(\theta) = \frac{\sin\left(n + \frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}. \tag{7.4.15}$$

Clearly, $D_n(\theta)$ is an even function with period 2π and satisfies the property

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(\theta)d\theta = 1 + 0 + 0 + \cdots + 0 = 1. \tag{7.4.16}$$

It is important to point out that the mean square convergence does not provide any insight into the problem of pointwise convergence. An infinite series $\sum_{n=1}^{\infty} f_n(x)$ is called *pointwise convergent* in $a < x < b$ to $f(x)$ if it converges to $f(x)$ for each x in $a < x < b$. In other words, for each x in $a < x < b$, we have

$$|f(x) - s_n(x)| \to 0 \quad \text{as} \quad n \to \infty, \tag{7.4.17}$$

where $s_n(x) = \sum_{k=1}^{n} f_k(x)$ is the nth partial sum of the infinite series.

On the other hand, the series $\sum_{k=1}^{\infty} f_k(x)$ is said to be *uniformly convergent* to $f(x)$ in $a \le x \le b$ if

$$\max_{a\le x\le b} |f(x) - s_n(x)| \to 0 \quad \text{as} \quad n \to \infty. \tag{7.4.18}$$

Evidently, uniform convergence implies pointwise convergence, but the converse is not necessarily true. It is also noted that uniform convergence is stronger than both pointwise convergence and mean-square convergence. Indeed, the mean-square convergence theorem does not guarantee the convergence of the Fourier series for any x. On the other hand, if $f(x)$ is 2π-periodic and piecewise smooth on $\mathbb{R}$, then the Fourier series (7.4.4) of the function f converges to $f(x)$ for every x in $-\pi \le x \le \pi$. It has been known since 1876 that there are periodic continuous functions whose Fourier series *diverge* at certain points. Many great mathematicians including Fourier, Riemann, Dirichlet, Georg Cantor, Paul Du Bois–Reymond (1831-1889) and Andrei N. Kolmogorov (1903-1987) paid considerable attention to the convergence problem of Fourier series and trigonometric series. In his famous paper of 1829, Dirichlet formulated a set of *sufficient conditions* that the Fourier series generated by a given $f(x)$ converges to $f(x)$. Under the guidance of Dirichlet, Riemann also investigated necessary and sufficient conditions that a function must satisfy so that at a point x in $-\pi \le x \le \pi$ the Fourier series for $f(x)$ should converge to $f(x)$. Although Riemann did prove the fundamental result that if $f(x)$ bounded and integrable in $-\pi \le x \le \pi$, then the Fourier coefficients a_n and b_n defined by (7.4.5) and (7.4.6) tend to zero as $n \to \infty$, but the convergence problem of Fourier series remained unsolved.

For about fifty years after Dirichlet's work, it was generally believed that the Fourier series for any continuous function $f(x)$ in $-\pi \le x \le \pi$

converges to $f(x)$. But in 1873, Du Bois–Reymond constructed an example of a function continuous in $-\pi \le x \le \pi$ whose Fourier series did not converge at a particular point. In 1883, he also showed that any Fourier series for a function $f(x)$ that is Riemann integrable can be integrated term by term, even though the series is not uniformly convergent. In 1893, Camille Jordan developed a sufficient condition in terms of a function of bounded variation which was introduced by him. His sufficient conditions assert that the Fourier series for an integrable function $f(x)$ converges to $\frac{1}{2}[f(x-0)+f(x+0)]$ at every point x for which there is a neighborhood in which $f(x)$ is of bounded variation. In 1926, Kolmogorov gave an example of a function $f \in L^1$ whose Fourier series diverges everywhere. It was an open question for a period of a century whether a Fourier series of a continuous function converges at any point. In 1966, Lennant Carleson provided an affirmative answer with a deep theorem which states that the Fourier series of any square integrable function $f(x)$ in $-\pi \le x \le \pi$ converges to $f(x)$ at almost every point.

We next state the *Pointwise Convergence Theorem.* If $f(x)$ is a piecewise smooth and periodic function in $-\pi \le x \le \pi$ with period 2π, then, for any x in $(-\pi,\pi)$, the Fourier series for $f(x)$ converges to $\frac{1}{2}[f(x-0)+f(x+0)]$. In other words,

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\frac{1}{2}[f(x-0)+f(x+0)], \qquad (7.4.19)$$

where x is any point of jump discontinuity in $(-\pi,\pi)$, and

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt dt, \quad n=0,1,2,3,\cdots \qquad (7.4.20)$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt dt, \quad n=0,1,2,3,\cdots. \qquad (7.4.21)$$

Obviously, at any point of continuity x in $[-\pi,\pi]$, the Fourier series (7.4.19) for $f(x)$ converges to $f(x)$, and at the endpoints, $x=\pm\pi$, the Fourier series (7.4.19) converges to $\frac{1}{2}[f(-\pi-0)+f(\pi+0)]$.

In his paper of 1754, Euler derived trigonometric series representations of a function in a totally different manner. He wrote the geometric series involving trigonometric functions in the form

$$\sum_{n=0}^{\infty} a^n(\cos x+i\sin x)^n=\frac{1}{1-a(\cos x+i\sin x)}. \qquad (7.4.22)$$

Or, equivalently,

$$\sum_{n=0}^{\infty} a^n(\cos nx+i\sin nx)=\frac{1}{1-a(\cos x+i\sin x)}.$$

Multiplying both numerator and denominator by the complex conjugate of the denominator of the right hand sides gives

$$1 + \sum_{n=1}^{\infty} a^n (\cos nx + i \sin nx) = \frac{(1 - a\cos x) - ia \sin x}{(1 - 2a\cos x + a^2)}.$$

Equating real and imaginary parts yields

$$\sum_{n=1}^{\infty} a^n \cos nx = \frac{a\cos x - a^2}{(1 - 2a\cos x + a^2)} \tag{7.4.23}$$

$$\sum_{n=1}^{\infty} a^n \sin nx = \frac{a\sin x}{(1 - 2a\cos x + a^2)}. \tag{7.4.24}$$

Substituting $a = \pm 1$ in (7.4.23) gives a divergent series

$$\frac{1}{2} = 1 \pm \cos x + \cos 2x \pm \sin 3x + \cos 4x \pm \cdots . \tag{7.4.25}$$

Euler then formally integrated this result to obtain the Fourier series for the algebraic function as

$$\frac{1}{2}(\pi - x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}, \qquad 0 < x < \pi, \tag{7.4.26}$$

and

$$\frac{x}{2} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin nx}{n}, \qquad -\pi < x < \pi. \tag{7.4.27}$$

Integrating term by term and determining the constant of integration gives

$$\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos nx}{n^2}. \tag{7.4.28}$$

During the 1860's, the properties of the Fourier coefficients were also investigated and among many important results obtained were what is called the *Parseval formula* proved formally by Marc-Antoine Parseval (1755-1836) in 1799. The Parseval formula can formally be derived from the convergence of Fourier series to $f(x)$ in $-\pi \le x \le \pi$. In other words, if (7.4.4) is a Fourier series of $f(x)$ with the Fourier coefficients (7.4.5) and (7.4.6), then the Parseval formula is given by

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right). \tag{7.4.29}$$

We multiply (7.4.4) by $\frac{1}{\pi}f(x)$ and then integrate the result from $-\pi$ to π to obtain

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{a_0}{2\pi}\int_{-\pi}^{\pi} f(x)dx + \sum_{n=1}^{\infty}\left[\frac{a_n}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx + \frac{b_n}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx\right]. \quad (7.4.30)$$

Replacing all integrals on the right hand side of (7.4.30) by the Fourier coefficients a_n and b_n gives the desired Parseval formula (7.4.29).

More generally, if $f(x)$ and $g(x)$ have convergent Fourier series in $-\pi \le x \le \pi$ with Fourier coefficients a_n, b_n and c_n, d_n respectively, then the following *generalized Parseval formula* holds

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx = \frac{a_0 c_0}{2} + \sum_{n=1}^{\infty}(a_n c_n + b_n d_n)\,. \quad (7.4.31)$$

When $f(x) = g(x)$, then (7.4.31) reduces to (7.4.29).

7.5 Generalized Zeta Function

Another *generalized zeta function* is defined by

$$\zeta(s,\nu) = \sum_{n=0}^{\infty}\frac{1}{(n+\nu)^s}, \quad \nu \neq 0, -1, -2, \cdots, \quad Re\, s > 1. \quad (7.5.1)$$

It can also be defined by the integral as

$$\zeta(s,\nu) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}e^{-\nu t}(1-e^{-t})^{-1}dt, \quad Re\, s > 1,\ Re\,\nu > 0. \quad (7.5.2)$$

When $\nu = 1$, the generalized zeta function (7.5.1) reduces to the ordinary zeta function (7.2.2), and the integral formula (7.5.2) reduces to (7.3.72). For more information and properties of the generalized zeta function, the reader is referred to Erdélyi et al. (1953, 1955).

We close this section by briefly mentioning the zeta function for more than one variable. Almost more than thirty years after his great discovery of the zeta function of one variable, Euler in 1775 introduced the zeta function of two variables s and r defined by

$$\zeta(s,r) = \sum_{n>m\geq 1}\frac{1}{n^s\, m^r}, \quad (7.5.3)$$

where $s \geq 2$ and $r \geq 1$. Euler simplified his definition by putting $n = m$ so that

$$\zeta^*(s,r) = \sum_{n \geq m \geq 1} \frac{1}{n^s\, m^r}, \tag{7.5.4}$$

so that

$$\zeta^*(s,r) = \zeta(s,r) + \zeta(s+r). \tag{7.5.5}$$

The product of two zeta functions $\zeta(s)$ and $\zeta(r)$ gives the identity

$$\zeta(s)\zeta(r) = \left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n^r}\right) = \zeta(s,r) + \zeta(r,s) + \zeta(s+r). \tag{7.5.6}$$

Combining (7.5.5) and (7.5.6) yields

$$\zeta(s)\zeta(r) = \zeta^*(s,r) + \zeta^*(r,s) - \zeta(r+s). \tag{7.5.7}$$

Euler also obtained the particular identity

$$\zeta(2,1) = \zeta(3) \tag{7.5.8}$$

and then a general identity

$$\zeta(s,1) + \zeta(s-1,2) + \cdots + \zeta(2,s-1) = \zeta(s+1), \tag{7.5.9}$$

$$2\zeta(s-1,1) - (s-1)\zeta(s) = \sum_{2 \leq r \leq s-2} \zeta(s)\zeta(s-r). \tag{7.5.10}$$

Making reference to Hoffman (1992) and Varadarajan (2007), we close this section by stating the definition of $\zeta\,(s_1, s_2, \cdots, s_k)$ of k variables as

$$\zeta\,(s_1, s_2, \cdots, s_k) = \sum_{n_1 > n_2 > \cdots n_k} \frac{1}{(n_1^{s_1}\, n_2^{s_2}\, \cdots\, n_k^{s_k})}, \tag{7.5.11}$$

where $s_r \geq 1$, $r = 1, 2, \cdots, k$.

A few identities for $\zeta(s_1, s_2, \cdots, s_k)$ have been proved. In recent years, the zeta function of several variables and their properties have received considerable attention with several conjectures. However, the progress is relatively slow.

7.6 Applications of the Zeta Function to Mathematical Physics and Algebraic Geometry

In the previous section, we have cited several remarkable applications of the zeta function to number theory and analysis. First, the Euler product formula played a central role for further understanding of the distribution of primes. Second, the zeta function has extensively been used to find the exact or approximate sum of many important numerical series. The zeros of the zeta function have also been closely associated with the eigenvalues of some random Hermitian matrix in quantum physics. David Hilbert once conjectured that the zeros of the zeta functions were distributed like the eigenvalues of certain types of random Hermitian matrix. In the physics of large nuclei, the eigenvalues of the same kind of matrix correspond to the energy levels of the nucleus (protons and neutrons). Conversely, random Hermitian matrices are successfully used to approximate the energy levels of large nucleii such as uranium, where the large number of protons and neutrons makes it almost impossible to explicitly model the nucleus.

Eugene Wigner (1902-1995) derived the distribution of energy level differences. In early 1970s, H. L. Montgomery derived a simple correlation function for two zeros of the zeta function and then Freeman Dyson recognized that this function is the two point correlation function for the distribution of eigenvalues of an $N \times N$ random Hermitian matrix given by

$$p(x) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2. \tag{7.6.1}$$

He also recognized that $\int_0^\alpha p(x)dx$ represents a function of α which can be used in modeling energy levels in quantum chaos. In 1973, based on extensive computations by Andrew Odlyzko at Bell Laboratory, it has also been shown that there is an excellent agreement between the distribution of the zeros of the zeta function and the eigenvalues of some matrix, and for higher order correlation functions. Thus, it seems that the zeros of the Riemann zeta function are represented by the eigenvalues of certain particular Hermitian matrix. This shows a kind of computational proof of Hilbert's conjecture made almost one hundred years ago.

In addition, the zeta function is used to represent a partition function in statistical physics. Given the energy states E_1, E_2, E_3, $\cdots$ of a number of particles, many physical properties of a statistical system can be described by the following partition function

$$P = \sum_{n=1}^{\infty} \exp\left(-\frac{E_n}{kT}\right), \tag{7.6.2}$$

where k is the Boltzmann constant, and T is the absolute temperature of the system. Setting $E_n = (kT)s \ln n$, $n = 1, 2, 3, \cdots$, it turns out that

$$P = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s). \tag{7.6.3}$$

Thus, the zeta function represents a particular partition function of a statistical system.

It is a well known fact that renormalization in physics is in some sense an extension of Euler's method of summation of divergent series. Divergent series often arises in some problems in statistical physics and quantum field theory. If a value is assigned to a divergent series in some sense, this value often agrees remarkably well with experimental observations. This strongly support Euler's method of determining a value of a divergent series so that the divergent series becomes a useful entity.

For example, the trace of an infinite identity matrix I is the sum of its all diagonal elements, so that

$$tr(I) = 1 + 1 + 1 + \cdots = \lim_{n\to\infty} n = \infty. \tag{7.6.4}$$

In order to assign a suitable value to the trace of I, we can use the zeta function defined by (7.2.2) so that $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ for a real or complex s. In particular, when $s = 0$, the value of the zeta function is

$$\zeta(0) = 1 + 1 + 1 + \cdots \infty. \tag{7.6.5}$$

Thus, the renormalized value of the trace of the matrix I can be defined by

$$tr(I)_{ren} = \zeta(0) = -\frac{1}{2}. \tag{7.6.6}$$

Thus, a negative number is assigned to a divergent infinite series in order to justify some experimental observations in physics. This is a very surprising result in mathematics and physics.

In 1940, André Weil proved that the zeta function of a smooth projective algebraic curve defined over a finite field satisfies the analogue of the Riemann Hypothesis. In 1949, he conjectured that such a result should be true for the zeta function of smooth projective varieties of any dimension defined over finite fields. Thus, the zeta function is found to occur in algebraic geometry and has a possible link in the solutions of some major problems in algebraic geometry.

Finally, we conclude this chapter by adding a very recent truely extraordinary breakthrough in the work of Green and Tao (2008) involving infinitely many primes in the arithematic progression (7.3.13). They proved that for every n, there are infinitely many n-term arithematic progressions of primes, that is, $a,\ a+h,\ \cdots,\ a+(n-1)h$ are all primes. They are also interested in finding an approximate formula for the number of n-term arithmetic progressions of primes and its extension to the number of primes in polynomials.

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Chapter 8

Euler's Beta and Gamma Functions and Infinite Products

> "No mathematician ever attained such a position of undisputed leadership in all branches of mathematics, pure and applied, as Euler did for the best part of the eighteenth century."
>
> *André Weil*

> "Of the so-called "higher mathematical functions," the gamma function is undoubtedly the most fundamental."
>
> *Phillip J. Davis*

8.1 Introduction

Historically, around 1729, Euler expanded $(1-x)^n$ by binomial theorem for an integer n and obtained

$$\int_0^1 x^m (1-x)^n dx = \frac{1.2.3.\cdots.n}{(m+1)(m+2)\cdots(m+n+1)}, \qquad (8.1.1)$$

where m is an arbitrary number. His idea was to isolate the product 1, 2, 3, $\cdots$, n from the denominator and then find an expression for $n!$ as an integral. With formal manipulation of (8.1.1), Euler derived an integral for $n!$ in the form

$$n! = \int_0^1 (-\log x)^n \, dx. \qquad (8.1.2)$$

Thus, the above two results (8.1.1) and (8.1.2) led Euler to discover the beta and the gamma functions and to study their basic properties. From the correspondence with Christian Goldback in 1729, Euler first generalized the factorial function and introduced the *Eulerian integral of the second kind* in 1730 which represents the *Euler gamma function*.

8.2 Euler's Beta and Gamma Functions

The *Eulerian integral of the first kind* represents the *Euler beta function*, $B(x, y)$ which is defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \qquad x > 0, \quad y > 0. \tag{8.2.1}$$

The beta function has many simple but interesting properties such as

$$B(x,y) = B(y,x), \tag{8.2.2}$$

$$B(x,y)B(x+y,z) = B(y,z)B(y+z,x). \tag{8.2.3}$$

Substituting $t = (1+s)^{-1}$ in (8.2.1) gives an alternative form of $B(x, y)$ by an infinite integral as

$$B(x,y) = \int_0^\infty \frac{s^{y-1}ds}{(1+s)^{x+y}} = \int_0^\infty \frac{s^{x-1}ds}{(1+s)^{x+y}}, \quad x > 0, \quad y > 0. \tag{8.2.4}$$

Putting $t = \cos^2\theta$ in (8.2.1) gives

$$B(x,y) = 2\int_0^{\pi/2} \cos^{2x-1}\theta\,\sin^{2y-1}\theta\,d\theta. \tag{8.2.5}$$

When m and n are nonnegative integers,

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n+1)!}. \tag{8.2.6}$$

Several important results are given below without proof.

$$B(1,1) = 1, \qquad B\left(\frac{1}{2},\frac{1}{2}\right) = \pi, \tag{8.2.7}$$

$$B(x,y) = \left(\frac{x-1}{x+y-1}\right) B(x-1,y), \tag{8.2.8}$$

$$B\left(\frac{1+x}{2},\frac{1-x}{2}\right) = \pi\sec\left(\frac{\pi x}{2}\right), \qquad 0 < x < 1. \tag{8.2.9}$$

Euler defined the gamma function, $\Gamma(z)$ by an infinite integral in the form

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \qquad Re\, z > 0. \tag{8.2.10}$$

Putting $e^{-t} = u$ gives an alternating form of $\Gamma(z)$ as

$$\Gamma(z) = \int_0^1 (-\ln u)^{z-1}\,du. \tag{8.2.11}$$

For integral values of $z = n+1$, result (8.2.11) gives

$$\Gamma\left(n+1\right) = \int_0^1 \left(-\ln t\right)^n dt. \tag{8.2.12}$$

It is important to note that $\Gamma(z)$ is analytic for $Re\, z > 0$ and appears frequently in expressions for the asymptotic analysis of differential equations. However, there is no algebraic differential equation for the gamma function which was proved by O. Hölder and hence, it is called a *transcendental function*. This is somewhat similar to the *transcendental numbers* e, π, γ which does not satisfy any algebraic equation.

Integrating (8.2.10) by parts yields the relation

$$\begin{aligned}\Gamma(z) &= \left[-e^{-t}t^{z-1}\right]_0^\infty + (z-1)\int_0^\infty e^{-t}t^{z-2}dt \\ &= (z-1)\Gamma(z-1).\end{aligned}$$

Replacing z by $z+1$ gives the fundamental recurrence relation

$$\Gamma(z+1) = z\,\Gamma(z). \tag{8.2.13}$$

Hence, for integral values of $z = n$ gives

$$\Gamma(n+1) = 1.2.3.\cdots.n = n! \tag{8.2.14}$$

This shows that the gamma function is a generalization of the factorial function. Further, the relation (8.2.13) can be used to continue analytically $\Gamma(z)$ to values of z for $Re\, z > 0$.

Another equivalent form of the gamma function for real $z = x$ can be obtained from (8.2.10) by the change of variable $t = u^2$ in the form

$$\Gamma(x) = 2\int_0^\infty \exp\left(-u^2\right) u^{2x-1}du, \qquad x > 0. \tag{8.2.15}$$

Putting $x = \frac{1}{2}$ in (8.2.15) leads to the following formula discovered by Euler in 1730

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}du = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}. \tag{8.2.16}$$

In his letter to Goldback in 1729, Euler gave another alternative definition of the gamma function in terms of a limit in the form

$$\Gamma(x) = \lim_{n\to\infty}\Gamma_n(x) = \lim_{n\to\infty}\frac{n!\,n^x}{x(x+1)\cdots(x+n)}. \tag{8.2.17}$$

It also easy to check that

$$\Gamma_n(x+1) = \frac{n}{(n+x+1)}\cdot x\,\Gamma_n(x). \tag{8.2.18}$$

There is a fundamental relation between the beta and the gamma function given by

$$B(x,y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma\,(x+y)}. \tag{8.2.19}$$

This follows from (8.2.10) that

$$\Gamma(x)\Gamma(y)=\int_0^\infty e^{-t}t^{x-1}dt\int_0^\infty e^{-s}s^{y-1}ds. \tag{8.2.20}$$

Also, $\Gamma(x)$ is uniformly convergent for all real x in $a\le x\le b$, where $0\le a\le x\le b<\infty$ and hence, $\Gamma(x)$ is a continuous function of x for all $x>0$.

Introducing polar coordinates by $\sqrt{t}=r\cos\theta$ and $\sqrt{s}=r\sin\theta$ with $\rho=r^2$, result (8.2.20) becomes

$$\Gamma(x)\Gamma(y)=2\int_0^\infty e^{-\rho}\rho^{x+y-1}d\rho\int_0^{\pi/2}\cos^{2x-1}\theta\,\sin^{2y-1}\theta\,d\theta.$$

Substituting $u=\sin^2\theta$ in the above integral and using (8.2.5) gives

$$\Gamma(x)\Gamma(y)=\Gamma\,(x+y)\,B(x,y).$$

Both the gamma and beta functions are related to the Laplace transform and its convolution property (for details, see Debnath and Bhatta (2007)). The Laplace transform of a function $f(t)$ is defined by

$$F(s)=\mathfrak{L}\{f(t)\}=\int_0^\infty e^{-st}f(t)dt,\qquad Re\,s>0. \tag{8.2.21}$$

Thus, it follows from (8.2.21) that

$$\mathfrak{L}\left\{t^{x-1}\right\}=\int_0^\infty e^{-st}t^{x-1}dt=\frac{\Gamma(x)}{s^x}. \tag{8.2.22}$$

On the other hand, the convolution of $f(t)$ and $g(t)$ is defined by

$$f(t)*g(t)=\int_0^t f\,(t-\tau)\,g(\tau)d\tau=\int_0^t g\,(t-\tau)\,g(\tau)d\tau. \tag{8.2.23}$$

The convolution theorem in the Laplace transform theory asserts that

$$\mathfrak{L}\{f(t)*g(t)\}=F(s)G(s)=\mathfrak{L}\{f(t)\}\,\mathfrak{L}\{g(t)\}\,. \tag{8.2.24}$$

Or equivalently,

$$f(t)*g(t)=\mathfrak{L}^{-1}\{F(s)G(s)\}\,. \tag{8.2.25}$$

So, it follows that

$$\int_0^t \tau^{x-1}\,(t-\tau)^{y-1}\,d\tau=t^{x-1}*t^{y-1} \tag{8.2.26}$$

where the integral in (8.2.22) defines the beta function $B(x,y)$ when $t=1$.

With $f(t)=t^{x-1}$, $g(t)=t^{y-1}$, we obtain

$$\mathfrak{L}\left\{t^{x-1}\right\}=\frac{\Gamma(x)}{s^x} \quad \text{and} \quad \mathfrak{L}\{g(t)\}=\frac{\Gamma(y)}{s^y}. \tag{8.2.27}$$

It follows from the Laplace convolution theorem that

$$\begin{aligned}\int_0^t \tau^{x-1}(t-\tau)^{y-1}d\tau = t^{x-1} * t^{y-1} &= \mathfrak{L}^{-1}\left\{\frac{\Gamma(x)}{s^x}\cdot\frac{\Gamma(y)}{s^y}\right\}\\ &= \Gamma(x)\Gamma(y)\mathfrak{L}^{-1}\left\{\frac{1}{s^{x+y}}\right\}\\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\,t^{x+y-1}.\end{aligned} \tag{8.2.28}$$

Putting $t=1$ in (8.2.28) leads to the result (8.2.19).

Euler also proved the following general *reflection formula*

$$B(x,\ 1-x)=\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x}, \qquad 0<x<1. \tag{8.2.29}$$

Using logarithm differentiation of (8.2.29) with respect to x gives

$$-\frac{\Gamma'(x)}{\Gamma(x)}+\frac{\Gamma'(1-x)}{\Gamma(1-x)}=\pi\cot\pi x. \tag{8.2.30}$$

In modern notation, the Euler definition of $\Gamma(1+m)$ for positive real number m is given by

$$\Gamma(1+m)=\lim_{n\to\infty}\frac{n!}{(m+1)(m+2)\cdots(m+n)}(n+1)^m. \tag{8.2.31}$$

He then wrote another form

$$\Gamma(1+m)=\lim_{n\to\infty}\frac{1.2^m}{(m+1)}\cdot\frac{2^{1-m}\,3^m}{(m+2)}\cdot\cdots\cdot\frac{n^{1-m}(n+1)^m}{(m+n)}. \tag{8.2.32}$$

The Euler universal constant γ is defined by

$$\gamma=\lim_{n\to\infty}\gamma_n=\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log(n+1)\right), \tag{8.2.33}$$

with the numerical value

$$\gamma=0.5772156649015329\cdots. \tag{8.2.34}$$

Since $\log(n+1)=\sum_{k=1}^{n}\log\left(\frac{k+1}{k}\right)$, we can write (8.2.33) as

$$\gamma=\lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)-\sum_{k=1}^{n}\log\left(\frac{k+1}{k}\right)\right].$$

Then

$$\lim_{n\to\infty} \exp\left[\left\{-z\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right\}+z\sum_{k=1}^{n}\log\left(\frac{k+1}{k}\right)\right]=e^{-\gamma z}. \quad (8.2.35)$$

Since

$$\exp\left[-z\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right]=\prod_{k=1}^{n}\exp\left(-\frac{z}{k}\right)$$

$$\text{and}\quad \exp\left[z\sum_{k=1}^{n}\log\left(\frac{k+1}{k}\right)\right]=\prod_{k=1}^{n}\left(\frac{k+1}{k}\right)^{z}$$

result (8.2.35) can be expressed in the form

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left[\left(1+\frac{1}{k}\right)^{z}\exp\left(-\frac{z}{k}\right)\right]=\exp\left(-\gamma z\right). \quad (8.2.36)$$

If $Re(z) > 0$, *Euler's infinite product representation* of $\Gamma(z)$ is

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right]. \quad (8.2.37)$$

To prove this result, we first show that

$$\lim_{n\to\infty}\frac{n!\,n^{z}}{z(z+1)(z+2)\cdots(z+n)}=\frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right]. \quad (8.2.38)$$

This follows from the fact that

$$\begin{aligned}&\frac{n!\,(n+1)^{z}}{(z+1)(z+2)\cdots(z+n)}\\ &\quad=\frac{n!}{(z+1)(z+2)\cdots(z+n)}\cdot\frac{2^{z}}{1^{z}}\cdot\frac{3^{z}}{2^{z}}\cdot\frac{4^{z}}{3^{z}}\cdots\frac{(n+1)^{z}}{n^{z}}\\ &\quad=\prod_{k=1}^{n}\left[\left(\frac{k}{z+k}\right)\cdot\left(\frac{k+1}{k}\right)^{z}\right]=\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1}\left(1+\frac{1}{k}\right)^{z}. \quad (8.2.39)\end{aligned}$$

Since

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{z}=1,$$

we take the limit (8.3.39) as $n\to\infty$ and divide by $z\neq 0$ so that (8.2.38) follows immediately.

We next show that

$$\Gamma(z)=\lim_{n\to\infty}\int_{0}^{n}\left(1-\frac{t}{n}\right)^{n}t^{z-1}dz. \quad (8.2.40)$$

Putting $t = nu$ in the integral in (8.2.40) gives

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dz = n^z \int_0^1 (1-u)^n u^{z-1} du. \tag{8.2.41}$$

Integrating the integral on the right hand side by parts yields the reduction formula

$$\int_0^1 (1-u)^n u^{z-1} du = \frac{n}{z} \int_0^1 (1-u)^{n-1} u^z du$$

which is, by iteration,

$$\begin{aligned} &= \frac{n}{z} \cdot \frac{n-1}{z+1} \cdot \frac{n-2}{z+2} \cdots \frac{1}{z+n-1} \int_0^1 u^{z+n-1} du \\ &= \frac{n!}{z(z+1)(z+2)\cdots(z+n)}. \end{aligned}$$

Consequently, result (8.2.41) becomes

$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)}$$

so that the limit of this as $n \to \infty$ is

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\cdots(z+n)}.$$

Using the definition of $\Gamma(z)$ and writing

$$\begin{aligned} &\left|\Gamma(z) - \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt\right| \\ &= \left|\int_0^\infty e^{-t} t^{z-1} dt - \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt\right| \\ &= \left|\lim_{n\to\infty} \left[\int_0^\infty e^{-t} t^{z-1} dt - \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt\right]\right| \\ &= \left|\lim_{n\to\infty} \left[\int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{z-1} dt + \int_n^\infty e^{-t} t^{z-1} dt\right]\right|, \end{aligned}$$

it turns out that both integrals tend to zero as $n \to \infty$.

Thus, the Euler formula (8.2.37) follows from (8.2.38) and then, we can write the formula as

$$\begin{aligned} z\,\Gamma(z) &= \lim_{n\to\infty} \prod_{k=1}^n \left[\left(1 + \frac{1}{k}\right)^z \left(1 + \frac{z}{k}\right)^{-1}\right] \\ &= \lim_{n\to\infty} \prod_{k=1}^n \left[\left(1 + \frac{1}{k}\right)^z \exp\left(-\frac{z}{k}\right) \left(1 + \frac{z}{k}\right)^{-1} \exp\left(\frac{z}{k}\right)\right] \end{aligned}$$

which is, by (8.2.36),

$$= \exp(-\gamma z) \lim_{n\to\infty} \prod_{k=1}^{n} \left[\left(1+\frac{z}{k}\right)^{-1} \exp\left(\frac{z}{k}\right)\right]. \tag{8.2.42}$$

Inverting each member of (8.2.42) yields the celebrated definition of $\{\Gamma(z)\}^{-1}$ due to Weierstrass in the form

$$\frac{1}{\Gamma(z)} = z \exp(\gamma z) \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) \exp\left(-\frac{z}{n}\right). \tag{8.2.43}$$

This is a famous example of a cannonical product formula of Weierstrass and it defines an entire function with simple zeros at $z = 0, -1, -2, \cdots$. So the limit in (8.2.38) exists for all $z \neq 0, -1, -2, \cdots$, and defines $\Gamma(z)$ as a meromorphic function with simple poles at $z = 0, -1, -2, \cdots$. Thus, result (8.2.43) gives the *Euler reflection formula* as

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right) = -\frac{z}{\pi} \sin \pi z. \tag{8.2.44}$$

Or, equivalently,

$$\Gamma(z)\Gamma(-z) = -\left(\frac{\pi}{z \sin \pi z}\right), \tag{8.2.45}$$

which is, due to (8.2.13),

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{8.2.46}$$

This is an equivalent form of the *Euler celebrated reflection formula* discovered by Euler in 1771 and can provide us with a shortcut formula for numerical computations. Writing (8.2.46) as $F(z) = \sin \pi z\, \Gamma(z)\Gamma(1-z) - \pi = 0$, it can be shown that $F(z) = 0$ throughout the complex plane. Thus, when $z = \frac{1}{2}$, $\Gamma^2\left(\frac{1}{2}\right) = \pi$ and hence, $\Gamma\left(\frac{1}{2}\right) = \pm\sqrt{\pi}$. But the integrand in the definition (8.2.10) of the gamma function is positive when $x = \frac{1}{2}$ and hence, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Using the result, we can compute the $\Gamma(x)$ at $x = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \cdots$.

The Weierstrass factorization formula (8.2.43) shows that $\{1/\Gamma(z)\}$ is zero for $z = 0, -1, -2, \cdots$, and was the starting point how functions other than polynomial can be factorized. There were other similar factorization formula such as the product identity for the sine function discovered by Euler in 1748:

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1-\frac{z^2}{n^2}\right). \tag{8.2.47}$$

Although the factorization of polynomials is mainly an algebraic problem, but the extension to other functions such as the sine or the cosine which have infinite number of zeros required for the systematic development of a theory of infinite products. In 1876, Weierstrass successfully developed an extensive theory of factorization of other functions which included as special cases of these well-known infinite products, as well as of doubly periodic elliptic functions.

Differentiating (8.2.10) with respect to z gives

$$\frac{d\Gamma(z)}{dz} = \Gamma'(z) = \int_0^\infty t^{z-1}\,(\log t)\,e^{-t}dt \tag{8.2.48}$$

so that it leads to

$$\Gamma'(1) = \int_0^\infty e^{-t}\,(\log t)\,dt = -\gamma, \tag{8.2.49}$$

where γ is the universal Euler constant given by (8.2.33).

The Euler gamma function is known to satisfy the *multiplication formula of Gauss and Legendre*

$$\Gamma\left(nz\right) = (2\pi)^{\left(\frac{1-n}{2}\right)}\, n^{nz-\frac{1}{2}} \prod_{k=0}^{n-1}\left(z+\frac{k}{n}\right), \qquad n = 2, 3, \cdots. \tag{8.2.50}$$

In particular, when $n = 2$, (8.2.50) gives the *Legendre duplication formula*

$$\Gamma\left(2z\right) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right). \tag{8.2.51}$$

Other formulas satisfied by the Γ-functions include

$$\frac{\Gamma(x)}{\Gamma\left(x+y\right)} = e^{\gamma y}\prod_{n=0}^{\infty}\left(1+\frac{y}{x+n}\right)\exp\left(-\frac{y}{n+1}\right), \tag{8.2.52}$$

$$e^{\gamma\psi(x)}\cdot\frac{\Gamma(x)}{\Gamma\left(x+y\right)} = \prod_{n=0}^{\infty}\left(1+\frac{y}{x+n}\right)\exp\left(-\frac{y}{x+n}\right), \tag{8.2.53}$$

where γ is given by (8.2.33) and $\psi(x)$ is the *logarithm derivative* of $\Gamma(x)$ given by

$$\psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{8.2.54}$$

The function $\psi(x)$ is now known as the *digamma* or *psi function* which satisfies the identity $\psi(x+1) = \psi(x)+\frac{1}{x}$. The function $\psi(z)$ is a meromorphic function with simple poles at $z = 0,\ -1,\ -2,\ \cdots$ whose series expansion follows from the general expansion due to a great Swedish mathematician, Gosta Mittag–Leffler (1846-1927) in the form

$$\psi(z) = -\gamma + (z-1)\sum_{n=0}^{\infty} \frac{1}{(n+1)(z+n)}. \tag{8.2.55}$$

In addition, we obtain

$$\exp\left(y\psi(x)\right) = \exp\left(-\gamma y - \frac{y}{x}\right)\prod_{n=1}^{\infty} \exp\left[\frac{xy}{n(n+x)}\right]. \tag{8.2.56}$$

We next use Laplace's method of asymptotic expansion to obtain the *James Stirling approximation formula* for the function, $\Gamma(x+1)$ when $x \gg 1$. We use the definition (8.2.10) and rewrite it in an integral form to apply Laplace's method as follows:

$$\Gamma(x+1) = \int_0^{\infty} e^{-t} t^x dt = \int_0^{\infty} \exp\left(x\log t - t\right) dt,$$

which is, due to $t = x\tau$,

$$= x^{1+x}\int_0^{\infty} \exp\left[x\left(\log\tau - \tau\right)\right] d\tau,$$

which is the appropriate form used in the Laplace method to find the asymptotic value of the integral for large x.

Writing $f(\tau) = \log\tau - \tau$, $f'(\tau) = \tau^{-1} - 1$ and $f''(\tau) = -\frac{1}{\tau^2} < 0$. So, $f(\tau)$ has a local maximum at $\tau = 1$. We next use the Taylor series expansion of $f(\tau)$ as

$$\log\tau - \tau = \log\left[1 + (\tau-1) - 1 - (\tau-1)\right] = -1 - \frac{1}{2}(\tau-1)^2 + \cdots,$$

for $|\tau - 1| < 1$, and extend the limits of integration to obtain

$$\Gamma(x+1) \sim x^{1+x}e^{-x}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}x\,\alpha^2\right) d\alpha \quad \text{as} \quad x \to \infty,$$

which is, due to $y = \alpha\sqrt{x}$,

$$\begin{aligned}\Gamma(x+1) &\sim x^{x+\frac{1}{2}}e^{-x}\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}y^2\right) dy \\ &= \sqrt{2\pi}\,x^{x+\frac{1}{2}}e^{-x} \quad \text{as} \quad x \to \infty.\end{aligned} \tag{8.2.57}$$

Or, equivalently, when $x = n$

$$n! \sim \sqrt{2\pi}\,n^{n+\frac{1}{2}}e^{-n}, \qquad n \to \infty. \tag{8.2.58}$$

This is the celebrated *Stirling formula* for the factorial function, and the asymptotic formula (8.2.57) provides a good approximation for the gamma function for $x \gg 2$. However, very accurate asymptotic approximations are more useful in modern numerical computing. We discuss the asymptotic expansion of the gamma using another famous Euler's result, the *Euler–Maclaurin Summation Formula.* In 1732, Euler stated the "*Euler–Maclaurin*" formula which was also independently discovered by Euler and Colin Maclaurin in the period of 1732-1742. For a function $f(x)$ with continuous derivative of all orders up to and including $(2m+2)$ in $0 \le x \le n$, then the sum $\sum_{k=0}^{n} f(k)$ is given by the *Euler–Maclaurin summation formula*

$$\sum_{k=0}^{n} f(k) = \int_0^n f(t)dt + \frac{1}{2}[f(0)+f(n)] + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(n) - f^{(2k-1)}(0)\right] + R_m, \quad (8.2.59)$$

where B_{2k} are the Bernoulli numbers, and the remainder term R_m is

$$R_m = \frac{1}{(2m+1)!}\int_0^n B_{2m+1}(t) f^{(2m+1)}(t)dt. \quad (8.2.60)$$

In its simplest form, this summation formula for a function $f(x)$ with a continuous derivative in $0 \le x \le n$ is

$$\sum_{k=0}^{n} f(k) = \int_0^n f(t)dt + \frac{1}{2}[f(0)+f(n)] + \int_0^n \left(t - [t] - \frac{1}{2}\right) f'(t)dt, \quad (8.2.61)$$

where $[t]$ is the greatest integer function.

With $f(x) = \ln x$, and $f^{(k)}(x) = (-1)^{k+1}(k-1)!\,x^{-k}$, the Euler–Maclaurin summation formula gives the asymptotic expansion for $\ln n!$:

$$\ln n! \sim \left(n + \frac{1}{2}\right)\ln n + \frac{1}{2}\ln(2\pi) - n + \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)} n^{-(2k+1)}. \quad (8.2.62)$$

Or, equivalently,

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}}\, e^{-n}\, e^{r(n)}, \quad (8.2.63)$$

where

$$r(n) = \frac{B_2}{2}\cdot\frac{1}{n} + \frac{B_4}{12}\cdot\frac{1}{n^3} + \cdots = \frac{1}{12}\frac{1}{n} - \frac{1}{360}\frac{1}{n^3} + \cdots. \quad (8.2.64)$$

Expanding the exponential $e^{r(n)}$ as a power series, result (8.2.62) gives

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}\left[1 + \frac{1}{12n} + \frac{1}{288}\cdot\frac{1}{n^2} - \frac{139}{51840}\cdot\frac{1}{n^3} + \cdots\right]. \quad (8.2.65)$$

This asymptotic expansion seems to be very useful for large n, but the series is not convergent because the Bernoulli numbers fluctuate very rapidly. However, (8.2.65) is valid for all $n = x > 0$ and hence, using $\Gamma(x) = \frac{1}{x}\Gamma(x+1)$, we obtain the asymptotic expansion of $\Gamma(x)$ for large x:

$$\Gamma(x) \sim \sqrt{2\pi}\, x^{x+\frac{1}{2}} e^{-x} \left[1 + \frac{1}{12}\cdot\frac{1}{x} + \frac{1}{288}\cdot\frac{1}{x^2} - \frac{139}{51840}\cdot\frac{1}{x^3} + \cdots\right]. \tag{8.2.66}$$

In recent years, considerable attention has been given to the asymptotic expansion of the gamma function and the rate of convergence by several authors including Lanczos (1964), Karatsuba (2001), Shi et al. (2006) and Schmelzer and Trefethen (2007). In particular, Karatsuba (2001) reported one famous asymptotic result due to Ramanujan for $x \geq 0$:

$$\Gamma(x+1) \sim \sqrt{\pi}\, x^x e^{-x} \left[8x^3 + 4x^2 + x + \frac{1}{30}\, h(x)\right]^{\frac{1}{6}}, \tag{8.2.67}$$

where $h(x)$ is an increasing function of x in $1 \leq x < \infty$ with $h(1) = \left(e^6/\pi^3\right) - 13 \sim 0.0112$ and $h(\infty) = \frac{1}{30}$.

Using Euler's definition (8.2.32) and putting $m = \frac{1}{2}$, we obtain

$$\Gamma\left(1 + \frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} = \sqrt{\frac{1\cdot 2}{(3/2)^2}\cdot\frac{2\cdot 3}{(5/2)^2}\cdot\frac{3\cdot 4}{(7/2)^2}\cdots}$$

$$\frac{\pi}{4} = \frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdots. \tag{8.2.68}$$

This is the *celebrated Wallis product formula* for $\frac{\pi}{4}$.

John Wallis discovered this formula in 1655 by a different method as part of his work on finding the area of a circle.

This formula can also be derived from the Euler infinite product (8.2.47) for the sine function. When $z = \frac{1}{2}$ is substituted in (8.2.47), we obtain

$$\frac{2}{\pi} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) = \prod_{n=1}^{\infty}\frac{(2n-1)(2n+1)}{(2n)(2n)}$$

$$= \frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots. \tag{8.2.69}$$

This can be rearranged to obtain the Wallis formula (8.2.68) for $\frac{\pi}{4}$.

With the aid of the Mellin transform, we can derive the full Stirling approximation of the gamma function of all orders together with three constants involving the zeta function. Thus, we next use the Euler product representation for the gamma function in the form

$$\Gamma(x+1) = e^{-\gamma x}\prod_{n=1}^{\infty}\left(1 + \frac{x}{n}\right)^{-1}\exp\left(\frac{x}{n}\right). \tag{8.2.70}$$

We next take logarithm both sides of (8.2.70) and expand the logarithm in powers of x to obtain

$$\ln\Gamma(x+1) = -\gamma x + \sum_{n=1}^{\infty}\left\{\frac{x}{n} + \sum_{s=1}^{\infty}\frac{1}{s}\left(-\frac{x}{n}\right)^s\right\}, \tag{8.2.71}$$

which is, by changing the order of summation,

$$\ln\Gamma(x+1) = -\gamma x + \sum_{s=2}^{\infty}\frac{1}{s}\,\zeta(s)\,(-x)^s. \tag{8.2.72}$$

Replacing the sum in (8.2.72) by the Mellin transform and then evaluating the Mellin integral by the theory of residues, we obtain the full stirling asymptotic expansion in the form

$$\ln\Gamma(x+1) \sim \left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \frac{1}{\pi}\sum_{m}(-1)^{\frac{1}{2}(m-1)}\,\frac{\Gamma(m)\zeta(m+1)}{(2\pi x)^m}, \tag{8.2.73}$$

where the summation is taken over only odd integers m, and $\zeta(m) \to 1$ for large m.

8.3 Applications of the Euler Gamma Functions

The equation of the n-dimensional sphere (or ball) of radius r with center at the origin is

$$x_1^2 + x_2^2 + \cdots + x_n^2 = r^2, \tag{8.3.1}$$

where $(x_1, x_2, \cdots, x_n)$ represents a point on the sphere and $n \geq 2$.

The volume, V_n and the surface area, S_n of a sphere of radius r in n-dimensional space, $\mathbb{R}^n$ can be expressed in terms of the gamma function (see Debnath and Bhatta (2007)) as

$$V_n = \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^n r^n}{\Gamma\left(1+\frac{n}{2}\right)}, \qquad S_n = \frac{2\left\{\Gamma\left(\frac{1}{2}\right)\right\}^n r^{n-1}}{\Gamma\left(\frac{n}{2}\right)}. \tag{8.3.2}$$

Evidently, $dV_n = S_n$. In particular, when $n = 2, 3, \cdots$, $V_2 = \pi r^2$, $S_2 = dV_2 = 2\pi r$, $V_3 = \frac{4}{3}\pi r^3$, $S_3 = dV_3 = 4\pi r^2$, $V_4 = \frac{1}{2}\pi^2 r^4$, $S_4 = dV_4 = 2\pi^2 r^3$, $V_5 = \frac{8}{15}\pi^2 r^5$, $S_5 = dV_5 = \frac{8}{3}\pi^2 r^4$, $\cdots$. The volume and the surface area of an even-dimensional unit sphere ($r = 1$) are given by

$$V_{2n} = \frac{\pi^n}{n!} \quad \text{and} \quad S_{2n} = 2\pi\cdot\frac{\pi^{n-1}}{(n-1)!}, \tag{8.3.3}$$

so that the sum of all volumes as well as the sum of all surface areas are

$$\sum_{n=1}^{\infty} V_{2n} = \sum_{n=1}^{\infty} \frac{\pi^n}{n!} = e^{\pi} \quad \text{and} \quad \sum_{n=1}^{\infty} S_{2n} = 2\pi \sum_{n=1}^{\infty} \frac{\pi^{n-1}}{(n-1)!} = 2\pi e^{\pi}. \tag{8.3.4}$$

The complex gamma function occurs in the solution of mathematical problems in atomic, molecular and quantum physics. In particular, the radial wave functions for positive energy states in a Coulomb field satisfy differential equations involving the Euler complex gamma function. The gamma functions also involve in formulas for scattering of charged particles, for the nuclear forces between protons, in Enrico Fermi's (1901-1954) approximation formula for the probability of β-radiation and in many other problems. In view of wide spread occurrence of the gamma function in pure and applied mathematics, and the existence of the classical Stirling's asymptotic formula for the gamma function, the modern emphasis is to develop more sophisticated numerical and asymptotic methods for accurate computation of gamma functions in the complex plane.

8.4 Euler's Contributions to Infinite Products

Euler's research on infinite products and infinite series started as early as 1730. Even though at that time the concept of convergence of series was not available, Euler showed a tremendous interest in the subject. Euler knew that there are an infinite number of roots $x = 0, \pm\pi, \pm 2\pi, \cdots, \pm n\pi, \cdots$ of the equation $\sin x = 0$ and factorized $\sin x$ as infinite product in the form

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right). \tag{8.4.1}$$

Taking logarithms gives, for $0 < x < \pi$,

$$(\log \sin x - \log x) = \sum_{n=1}^{\infty} \left[\log\left(1 - \frac{x}{n\pi}\right) + \log\left(1 + \frac{x}{n\pi}\right)\right]. \tag{8.4.2}$$

Differentiating and then replacing x by πx leads to the remarkable partial fraction expansion

$$\pi \cot \pi x = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{n+x} - \frac{1}{n-x}\right), \qquad 0 < x < 1. \tag{8.4.3}$$

He also knew that $\sin x$, $\cos x$ have a power series representations

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots . \tag{8.4.4}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \infty. \tag{8.4.5}$$

Equating the coefficients of x^3 and x^5 in equations (8.4.1) and (8.4.4), Euler derived the formulas for numerical series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}, \qquad 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}. \tag{8.4.6ab}$$

He generalized these results to obtain the general series (7.2.2) to define the zeta function $\zeta(s)$ in around 1740.

We put $x = \pi/2$ on both sides of (8.4.1) to obtain

$$\begin{aligned}\frac{2}{\pi} &= \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) = \prod_{n=1}^{\infty}\frac{(4n^2-1)}{4n^2} \\ &= \prod_{n=1}^{\infty}\frac{(2n-1)(2n+1)}{(2n)(2n)} = \frac{1.3}{2.2}\cdot\frac{3.5}{4.4}\cdot\frac{5.7}{6.6}\cdot\frac{7.9}{8.8}\cdots .\end{aligned}$$

This can be reorganized in the form

$$\frac{\pi}{2} = \frac{2.2}{1.3}\cdot\frac{4.4}{3.5}\cdot\frac{6.6}{5.7}\cdot\frac{8.8}{7.9}\cdots . \tag{8.4.7}$$

This is one of the celebrated Wallis product formulas for $\frac{\pi}{2}$ which was proved by the British mathematician John Wallis in 1655 using a different method as part of his work on finding the area of a circle. However, in the *Introductio*, Euler discovered the Wallis formula (8.4.7) from the infinite product representation (8.4.1) of $\sin x$. One of the great mathematician, André Weil said that this is "One of Euler's most sensational early discoveries, perhaps the one which established his growing reputation most firmly".

Similarly, $\cos x$ can be written as an infinite product in the form

$$\cos x = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{\left[(2n-1)\frac{\pi}{2}\right]^2}\right). \tag{8.4.8}$$

Euler obtained the infinite product expansion of $\sin x$ and $\cos x$ from the identity

$$(z^{2m} - 1) = m(z^2 - 1)\prod_{n=1}^{m-1}\left[\frac{1 - 2z\cos\frac{n\pi}{m} + z^2}{2 - 2\cos\left(\frac{n\pi}{m}\right)}\right]. \tag{8.4.9}$$

Putting $z = \left(1 + \frac{x}{m}\right)$ in result (8.4.9) becomes

$$\left(1+\frac{x}{m}\right)^{m} - \left(1+\frac{x}{m}\right)^{-m} = \frac{\left(2x + \frac{x^2}{m}\right)}{\left(1+\frac{x}{m}\right)} \prod_{n=1}^{m-1} \left[1 + \frac{x^2}{\left(1+\frac{x}{m}\right)\left(2m \sin\frac{n\pi}{2m}\right)^2}\right]. \tag{8.4.10}$$

In the limit as $m \to \infty$, this reduces to

$$\frac{1}{2}(e^x - e^{-x}) = x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2\pi^2}\right). \tag{8.4.11}$$

Or

$$\sinh x = x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{n^2\pi^2}\right). \tag{8.4.12}$$

Replacing x by ix in (8.4.8) gives the product formula

$$\cosh x = \prod_{n=1}^{\infty} \left(1 + \frac{4x^2}{(2n-1)^2\pi^2}\right). \tag{8.4.13}$$

All of the above results remain valid for a complex variable $z = x + iy$. Weierstrass proved a remarkable theorem which guarantees the existence of infinite product expansions of a rather broad class of analytic functions of z. It has also been shown by him that the divergent infinite product

$$z\left(1+\frac{z}{\pi}\right)\left(1+\frac{z}{2\pi}\right)\left(1+\frac{z}{3\pi}\right)\cdots \tag{8.4.14}$$

can be made convergent by multiplying each factor by an exponential factor. Thus, the infinite product

$$z\left[\left\{\left(1+\frac{z}{\pi}\right)\exp^{-\frac{z}{\pi}}\right\}\left\{\left(1+\frac{z}{2\pi}\right)\exp^{-\frac{z}{2\pi}}\right\}\left\{\left(1+\frac{z}{3\pi}\right)\exp^{-\frac{z}{3\pi}}\right\}\cdots\right] \tag{8.4.15}$$

is absolutely convergent.

If $f(z)$ denotes the limit of the absolutely convergent product

$$\prod_{1}^{\infty}\left(1+\frac{z}{n\pi}\right)\exp\left(-\frac{z}{n\pi}\right),$$

and $f(-z)$ that of

$$\prod_{1}^{\infty}\left(1-\frac{z}{n\pi}\right)\exp\left(\frac{z}{n\pi}\right),$$

then

$$f(z)f(-z) = \prod_{1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right) = \frac{\sin z}{z}. \tag{8.4.16}$$

Thus, it follows that

$$\log \sin z = \log z + \sum_{n=1}^{\infty} \log\left(1 - \frac{z^2}{n^2\pi^2}\right).$$

Differentiating this result with respect to z yields

$$\cot z = \frac{1}{z} + \left(\frac{1}{z+\pi} + \frac{1}{z-\pi}\right) + \left(\frac{1}{z+2\pi} + \frac{1}{z-2\pi}\right) + \cdots, \tag{8.4.17}$$

$$= \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{1}{(z^2 - n^2\pi^2)}. \tag{8.4.18}$$

The series in (8.4.17) is semi-convergent, and that in (8.4.18) is absolutely convergent except for $z = 0, \pm\pi, \pm 2\pi, \cdots$ for which values of the series is divergent.

Similarly, it follows from differentiation of logartithms of the infinite product formula (8.4.8) for $\cos z$

$$\cos z = \left(1 - \frac{4z^2}{\pi^2}\right)\left(1 - \frac{4z^2}{3^2\pi^2}\right)\left(1 - \frac{4z^2}{5^2\pi^2}\right)\cdots, \tag{8.4.19}$$

that

$$-\tan z = \left\{\frac{1}{z+\frac{\pi}{2}} + \frac{1}{z-\frac{\pi}{2}}\right\} + \left\{\frac{1}{z+\frac{3\pi}{2}} + \frac{1}{z-\frac{3\pi}{2}}\right\} + \cdots, \tag{8.4.20}$$

or,

$$\tan z = 8z\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2\pi^2 - 4z^2}. \tag{8.4.21}$$

The series (8.4.20) is semi-convergent, but (8.4.21) is absolutely convergent for the values of z except for $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \cdots$.

Using the formula $\operatorname{cosec} z = \cot\frac{1}{2}z - \cot z$ or $\operatorname{cosec} z = \frac{1}{2}\cot z + \frac{1}{2}\tan\frac{1}{2}z$ and the series for the cotangents gives

$$\begin{aligned}\operatorname{cosec} z = &\left(\frac{2}{z} + \frac{2}{z+2\pi} + \frac{2}{z-2\pi} + \frac{2}{z+4\pi} + \frac{2}{z-4\pi} + \cdots\right)\\ &-\left(\frac{1}{z} + \frac{1}{z+\pi} + \frac{1}{z-\pi} + \frac{1}{z+2\pi} + \frac{1}{z-2\pi} + \cdots\right)\end{aligned} \tag{8.4.22}$$

$$= \frac{1}{z} - \frac{1}{z+\pi} - \frac{1}{z-\pi} + \frac{1}{z+2\pi} + \frac{1}{z-2\pi} - \frac{1}{z+3\pi} - \frac{1}{z-3\pi} + \cdots. \tag{8.4.23}$$

Or,

$$\operatorname{cosec} z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n 2z}{(z^2 - n^2\pi^2)}. \tag{8.4.24}$$

Replacing z by $(z + \frac{1}{2}\pi)$ in (8.4.23) gives

$$\sec z = \left(\frac{1}{z + \frac{\pi}{2}} - \frac{1}{z - \frac{\pi}{2}}\right) - \left(\frac{1}{z + \frac{3\pi}{2}} - \frac{1}{z - \frac{3\pi}{2}}\right) + \cdots. \tag{8.4.25}$$

Or,

$$\sec z = 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-1)\pi}{(2n-1)^2\pi^2 - 4z^2}. \tag{8.4.26}$$

When n is large, the series (8.4.26) has its general term tending to the value $(-1)^{n-1}(2n-1)^{-1}$, hence the series is only semi-convergent.

Taking logarithms of the product formula (8.4.1) for $\sin z$ and (8.4.19) for $\cos z$ yields

$$\log \sin z = \log z + \log\left(1 - \frac{z^2}{\pi^2}\right) + \log\left(1 - \frac{z^2}{2^2\pi^2}\right) + \cdots \tag{8.4.27}$$

$$\log \cos z = \log\left(1 - \frac{4z^2}{\pi^2}\right) + \log\left(1 - \frac{4z^2}{3^2\pi^2}\right) + \log\left(1 - \frac{4z^2}{5^2\pi^2}\right) + \cdots. \tag{8.4.28}$$

Expanding the logarithms in (8.4.27) for $|z| < \pi$ and in (8.4.28) for $z < \frac{\pi}{2}$ gives

$$\log\left(\frac{\sin z}{z}\right) = -\sum_{n=0}^{\infty}\left(\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \cdots\right)\frac{z^{2n}}{n\pi^{2n}}, \tag{8.4.29}$$

$$\log \cos z = -\sum_{n=0}^{\infty}\left(\frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \cdots\right)\frac{2^{2n}z^{2n}}{n\pi^{2n}}. \tag{8.4.30}$$

In view of the fact that

$$\begin{aligned}\zeta(2n) &= \frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots \\ &= \left(\frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \cdots\right) + \frac{1}{2^{2n}}\left(\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots\right)\end{aligned} \tag{8.4.31}$$

so that

$$\frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \cdots = \frac{2^{2n}-1}{2^{2n}}\,\zeta(2n). \tag{8.4.32}$$

Consequently,

$$\log\left(\frac{\sin z}{z}\right) = -\sum_{n=0}^{\infty} \frac{\zeta(2n)}{n\pi^{2n}}\, z^{2n} \tag{8.4.33}$$

$$\log\cos z = -\sum_{n=0}^{\infty} \frac{(2^{2n}-1)}{n\pi^{2n}}\, \zeta(2n)\, z^{2n}. \tag{8.4.34}$$

Using the result (7.3.64), these results reduce to

$$\log\left(\frac{\sin z}{z}\right) = -\left[2\,\frac{B_1}{1}\frac{z^2}{2!} + 2^3\,\frac{B_2}{2}\frac{z^4}{4!} + \cdots + 2^{2n-1}\frac{B_n}{n}\frac{z^{2n}}{(2n)!} + \cdots\right] \tag{8.4.35}$$

$$\log\cos z = -\left[2\,\frac{B_1}{1}\frac{z^2}{2!} + 2^3\,\frac{B_2}{2}\frac{z^4}{4!} + \cdots + 2^{2n-1}\frac{B_n}{n}\frac{z^{2n}}{(2n)!} + \cdots\right]. \tag{8.4.36}$$

Replacing the Bernoulli numbers by their numerical values given by (7.3.57) and (7.3.58) gives the series for the logarithmic sine and cosine in the form

$$\log\left(\frac{\sin z}{z}\right) = -\left(\frac{z^2}{6} + \frac{z^4}{180} + \frac{z^8}{2835} + \cdots\right). \tag{8.4.37}$$

$$\log\cos z = -\left(\frac{z^2}{2} + \frac{z^4}{12} + \frac{z^8}{45} + \cdots\right). \tag{8.4.38}$$

And hence,

$$\log\tan z = \log z + \left(\frac{z^2}{3} + \frac{7z^4}{30} + \frac{62z^6}{2835} + \cdots\right). \tag{8.4.39}$$

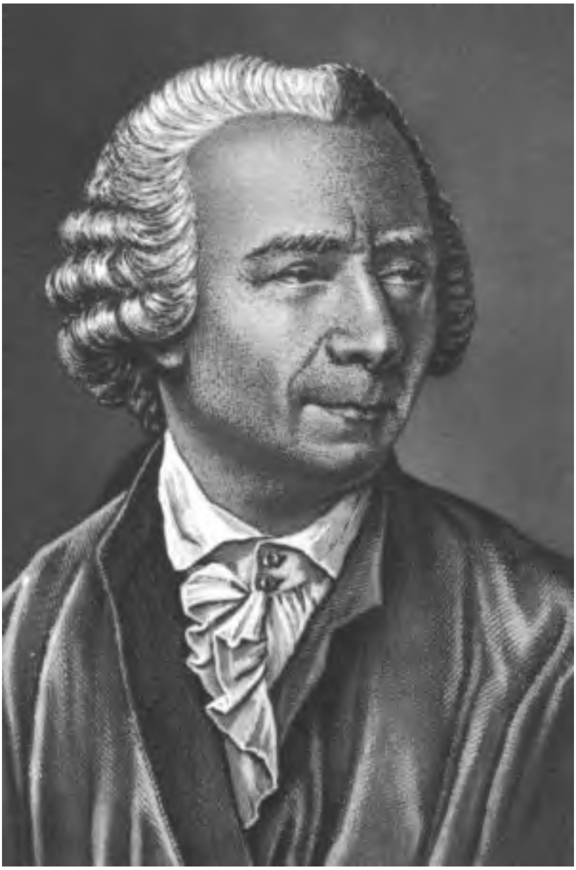
Euler substituted $z = \left(\frac{m\pi}{2n}\right)$ in (8.4.27) and (8.4.28) and then used the following resulting numerical series to calculate logarithmic sines and cosines upto twenty decimal places:

$$\log\sin\left(\frac{m\pi}{2n}\right) = \log\left(\frac{m\pi}{2n}\right) + \log\left(1 - \frac{m^2}{2^2\cdot n^2}\right) + \log\left(1 - \frac{m^2}{4^2\cdot n^2}\right) + \cdots \tag{8.4.40}$$

$$\log\cos\left(\frac{m\pi}{2n}\right) = \log\left(1 - \frac{m^2}{1^2\cdot n^2}\right) + \log\left(1 - \frac{m^2}{3^2\cdot n^2}\right) + \cdots. \tag{8.4.41}$$

He also obtained the following beautiful infinite products over only primes $p = 2, 3, 5, \cdots$:

$$\prod_{p=2}^{\infty}\left(1 - \frac{1}{p^2}\right) = \frac{6}{\pi^2},\ \prod_{p=2}^{\infty}\left(\frac{p^2}{p^2+1}\right) = \frac{\pi^2}{15},\ \prod_{p=2}^{\infty}\left(\frac{p^4}{p^4+1}\right) = \frac{\pi^4}{105}. \tag{8.4.42}$$



Chapter 9

Euler and Differential Equations

> "As the construction of the universe is the most perfect possible, being the handiwork of an all-wise Maker, nothing can be met with in the world in which some maximal and minimal property is not displayed. There is, consequently, no doubt but that all the effects of the world can be derived by the method of maxima and minima from their final causes as well as from their efficient ones."

Leonhard Euler

> "No mathematician ever attained such a position of undisputed leadership in all branches of mathematics, pure and applied, as Euler did for the best part of the eighteenth century."

André Weil

9.1 Historical Introduction

Historically, differential equations originated immediately after the independent discovery of calculus by Sir Isaac Newton and Gottfried W. Leibniz in the seventeenth century. In his famous 1953 book *Ordinary Differential Equations*, a British mathematician, Edward Lindsay Ince (1891-1941) stated the first discovery of solution of an ordinary differential equation by Leibniz in 1675:

"Yet our hazy knowledge of the birth and infancy of the science of differential equations condenses upon a remarkable date, the eleventh day of November, 1675, when Leibniz first set down on paper the equation

$$\int y dy = \frac{1}{2} y^2,$$

thereby not merely solving a simple differential equation, which was in itself a trivial matter, but what was an act of great moment, forging a powerful tool, the integral sign.

The early history of the infinitesimal calculus abounds in instances of problems solved through the agency of what were virtually differential equations; it is even true to say that the problem of integration, which may be regarded as the solution of the simplest of all types of differential equations, was a practical problem even in the middle of the sixteenth century. Particular cases of the inverse problem of tangents, that is the problem of determining a curve whose tangents are subjected to a particular law, were successfully dealt with before the invention of the calculus."

Although Newton did relatively little work in the theory of differential equations, he solved some ordinary differential equations in analytical form in his *Method of Fluxions* of 1671. In fact, he classified ordinary differential equations of the first order, then known as *fluxional equations* into three classes:

$$\text{(i)} \qquad \frac{\dot{y}}{\dot{x}} = f(x) \quad \text{or} \quad \frac{\dot{y}}{\dot{x}} = f(y), \tag{9.1.1}$$

where $\dot{x}$ and $\dot{y}$ represent *fluxions* (*or the rate of change of variables*) of *fluents* (*variables*) of x and y respectively.

$$\text{(ii)} \qquad \frac{\dot{y}}{\dot{x}} = f(x, y). \tag{9.1.2}$$

The third class consisted of equations involving more than two variables and they are now known as *partial differential equations*.

He also considered a large number of applications of fluxions to differentiating implicit functions and to determining tangents of plane curves, maxima and minima of functions, curvature of plane curves, and points of inflections of plane curves. He also first derived the precise formula for the radius of curvature, $\rho(x)$ of a plane curve $y = f(x)$ as

$$\rho(x) = \frac{1}{\kappa(x)} = \frac{(1 + \dot{y}^2)^{3/2}}{\ddot{y}}. \tag{9.1.3}$$

At the same time, Newton obtained lengths of plane curves and areas of closed curves. In mechanics, he formulated his law of falling body by the first order ordinary differential equation for the velocity v in a resisting medium and the law of cooling/heating for the temperature distribution $T(t)$ of a function of time.

On the other hand, Leibniz first introduced the modern notation for the first derivative, (dy/dx) or y' and the integral sign. In 1691, he first introduced a new variable $v = (y/x)$ to solve a class of the first order ordinary homogeneous equation of the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), \tag{9.1.4}$$

so that this equation reduces to a separable form. Indeed, he discovered the method of separation of variables to solve first order homogeneous equations and first order linear equations. In 1694, he showed how to reduce a linear first order ordinary differential equation form

$$y' + p(x)y = q(x) \tag{9.1.5}$$

to quadratures.

Following the work of Newton and Leibniz, the Bernoulli brothers Jakob and Johann and Johann's son, Daniel Bernoulli made some significant contributions to differential equations with applications to mechanics. In 1695, Johann Bernoulli first solved the first order nonlinear ordinary differential equation what is now known as the *Bernoulli equation*

$$\frac{dy}{dx} + p(x)y = q(x)\,y^n, \quad n \neq 0,\, 1. \tag{9.1.6}$$

However, it was Leibniz in 1696 who showed that equation (9.1.6) can be reduced to a linear equation by the substitution $v = y^{1-n}$. The Bernoulli brothers formulated the *brachistochrone problem* dealing with the determination of the curve of fastest descent in 1696 and reduced the problem to the first order nonlinear equation

$$y(1 + y'^2) = C, \tag{9.1.7}$$

where C is a constant. Newton also solved this problem in 1697.

In 1696, Johann Bernoulli considered the isoperimeter problem or the problem of determining plane curves of a given perimeter which enclose a maximum area. In 1698, he also considered the problem of determining a family of curves (*orthogonal trajectories*) which is orthogonal to a given family of curves. In a letter to Leibniz in 1716, Johann Bernoulli formulated a second order ordinary differential equation in the form

$$\frac{d^2y}{dx^2} = \frac{2y}{x^2}, \tag{9.1.8}$$

and showed that the solutions represent three types curves including parabolas, hyperbolas and curves of the third degree. In around 1724, an Italian mathematician, Count Jacopo Riccati (1676-1754) made some important contributions to differential equations in the form

$$F(y, y', y'') = F\left(y, p, p\,\frac{dp}{dy}\right) = 0, \tag{9.1.9}$$

where $p = y'$ so that the second order equation can be reduced to the first order in p. In the early history of ordinary differential equations, the first order nonlinear equation, now called the *Riccati equation*,

$$\frac{dy}{dx} = a_0(x) + a_1(x)\,y + a_2(x)\,y^2, \tag{9.1.10}$$

received a great deal attention. If $y = y_1$ is a particular solution, the general solution containing one arbitrary constant can be obtained through the substitution $y = y_1(x) + \frac{1}{v(x)}$, where $v(x)$ satisfies the first order linear equation

$$\frac{dv}{dx} + [2y_1 a_2(x) + a_1(x)]v = -a_2(x). \tag{9.1.11}$$

In particular, when $a_2(x) = -1$, the Riccati equation (9.1.10) becomes

$$\frac{dy}{dx} = a_0(x) + a_1(x)y - y^2. \tag{9.1.12}$$

The substitution $y = (u'/u)$ reduces (9.1.12) to the second-order linear equation

$$u'' - a_1(x)u' - a_0(x)u = 0. \tag{9.1.13}$$

The Riccati equation in the form

$$\frac{dy}{dx} + a_1(x)y + a_2(x)y^2 = 0 \tag{9.1.14}$$

can be reduced by the transformation $y = u^{-1}$ to the form

$$\frac{du}{dx} - a_1(x)u = a_2(x). \tag{9.1.15}$$

The general solution of this equation is

$$u(x) = Ce^{\int a_1(x)dx} + e^{\int a_1(x)dx}\left[\int^x a_2(t)\exp\left\{-\int a_1(t)dt\right\}dt\right], \tag{9.1.16}$$

where C is an arbitrary constant.

A British mathematician, Brook Taylor and a French mathematician, Alexis Claude Clairaut discovered singular solutions of a class of ordinary differential equations. A singular solution is truly a solution of the equation, but it cannot be derived from the general solution by assigning a particular value of the arbitrary constant involved in the general solution. In 1734, Clairaut considered the differential equation, known as the *Clairaut equation* in the form

$$y = px + f(p), \qquad p = y'. \tag{9.1.17}$$

He obtained the general solution by differentiating (9.1.17) with respect to x so that

$$p = p + \{x + f'(p)\} \frac{dp}{dx}. \tag{9.1.18}$$

This means that either

$$\frac{dp}{dx} = 0 \quad \text{or} \quad x + f'(p) = 0$$

so that the first equation leads to $p = y' = m$ and from the original equation we have

$$y = mx + f(m), \tag{9.1.19}$$

where m is an arbitrary constant. Thus, the Clairaut equation has the general solution (9.1.19) which represents a family of straight lines. The second equation $x + f'(p) = 0$ may be used together with the original equation (9.1.18) to eliminate p so that it gives a *new solution*, which is called the *singular solution* representing the envelope of the general solution (9.1.19). Differentiating (9.1.19) with respect m leads to

$$x + f'(m) = 0. \tag{9.1.20}$$

Thus, eliminating m from (9.1.19) and (9.1.20) leads to a plane curve which is the *envelope*. These two equations are exactly the same as the two equations that gives the singular solution.

For example, the general and singular solutions of the differential equation

$$y = p\,x + \frac{p}{a}, \qquad p = y' \tag{9.1.21}$$

are $y = mx + \frac{m}{a}$ and $y^2 = 4ax$ respectively and those of

$$y = px + \sqrt{a^2p^2 + b^2}, \qquad p = y' \tag{9.1.22}$$

are $y = mx + \sqrt{a^2m^2 + b^2}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ respectively, where m is an arbitrary constant.

Similarly, the differential equation associated with a plane curve such that the coordinate axes cut off from any tangent a constant length a is

$$(1 + p^2)(y - xp)^2 = a^2p^2, \qquad p = y'. \tag{9.1.23}$$

The singular solution of this equation is the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

The nonlinear ordinary differential equation of the first order

$$y'^2 + axy' + by + cx^2 = 0, \tag{9.1.24}$$

was investigated by G. Chrystal (1851-1911) in 1896. The parabola, $4by = -abx^2$ is a singular solution provided the coefficients of (9.1.24) satisfy the condition $a^2 + ab - 4c = 0$.

Both Euler and Clairaut gave a method of finding the singular solution from the differential equation itself, that is, by eliminating m from the following equations

$$f(x, y, m) = 0, \qquad \frac{\partial}{\partial m} f(x, y, m) = 0. \tag{9.1.25}$$

Euler was somewhat puzzled by the fact that singular solutions are not included in the general solution. In his *Institutiones* of 1768, Euler formulated a criterion for distinguishing the singular solution from a particular integral. Subsequently, considerable attention has been given to the problem of the existence of singular solutions of ordinary differential equations of the first order after the discovery and the basic work of Clairaut and Euler.

Several other mathematicians including d'Alembert, Arthur Cayley, J. G. Darboux and Lagrange made a systematic study of singular solutions and their relationship with the general solution. Lagrange gave the geometrical interpretation of the singular solution as the envelope of the family of integral curves.

Since there exist a number of references where the reader may find an adequate discussion of the theory of singular solutions, we simply review a few salient features of the subject. We consider a first order differential equation of the form

$$f(x, y, p) = 0, \qquad p = y'. \tag{9.1.26}$$

Then the partial derivative of this equation with respect to p gives the equation

$$f_p(x, y, p) = 0. \tag{9.1.27}$$

These two equations (9.1.26) and (9.1.27) form a system from which p can be eliminated in many cases so that the resulting equation $F(x, y) = 0$ can be found to define the p-*discriminant locus*.

Differentiating (9.1.26) with respect to x gives

$$f_x + p\, f_y + f_p \frac{dp}{dx} = 0, \tag{9.1.28}$$

which is, by (9.1.27)

$$f_x + p\, f_y = 0. \tag{9.1.29}$$

Thus, a necessary condition for the existence of a singular solution of (9.1.26) is given by the set of three equations

$$f = 0, \qquad f_p = 0, \qquad f_x + p\, f_y = 0. \tag{9.1.30}$$

Eliminating p from (9.1.30) yields the singular solution $F(x, y) = 0$.

Thus, the three equations (9.1.30) together with the condition $f_y \neq 0$ are sufficient for the existence of a singular solution of (9.1.26).

Hence, the singular solution of Chrystal's equation (9.1.24) must satisfy (9.1.30), that is,

$$p^2 + apx + by + cx^2 = 0, \quad 2p + ax = 0, \quad ap + 2cx + bp = 0. \tag{9.1.31}$$

The first two equations of (9.1.31) gives

$$4by = (a^2 - 4c)x^2, \tag{9.1.32}$$

and the third equation in (9.1.31) is satisfied provided $a^2 + ab = 4c$ which reduces (9.1.32) to the singular solution

$$4by = -abx^2 \tag{9.1.33}$$

provided $b \neq 0$ which is the condition $f_y \neq 0$. If $b = 0$, the solution of the original equation (9.1.24) degenerates into the parabola, $4y = -ax^2$, and there is *no* singular solution.

This chapter deals with major contributions of Euler to ordinary and partial differential equations. This is followed by the calculus of variations which is a new branch of mathematics effectively created by Euler in his 1744 famous book entitled *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes.*

9.2 Euler's Contributions to Ordinary Differential Equations

Euler made many significant contributions to the theory of ordinary differential equations and developed new analytical and numerical methods of solving differential equations with constant and variable coefficients. Many of his papers on differential equations dealt with physical problems. Of particular interest is his major work on the formulation of problems in mechanics in terms of differential equations and his development of mathematical methods of solving these equations. Lagrange once said of Euler's contributions to mechanics: "The first great work in which analysis is applied to the science of movement."

Euler first introduced the concept of an integrating factor and suggested a general treatment of linear ordinary differential equations with constant coefficients. He solved a class of first order linear equations in the form

$$\frac{dy}{dx} + p(x)y = q(x), \tag{9.2.1}$$

by introducing the integrating factor

$$\mu = \exp\left[\int p(x)dx\right]. \tag{9.2.2}$$

Euler also obtained the general solution $F(x, y) = C$ representing an implicit family of plane curves, where C is an arbitrary constant of the first order *exact* ordinary differential equations in the form

$$M(x, y)dx + N(x, y)dy = 0. \tag{9.2.3}$$

A necessary and sufficient condition that equation (9.2.3) is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

If equation (9.2.3) is *not* exact, then it is necessary to find a function $\mu(x, y)$ such that the expression

$$\mu\,[M(x, y)dx + N(x, y)dy] \tag{9.2.4}$$

is a total differential $d\,F(x, y)$. When μ has been found, the problem reduces to a mere quadrature.

The major question which arises is as to whether or not the integrating factor exists. It can be proved that under the assumption that the equation itself has one and only one solution which depends on an arbitrary constant, there exists an infinity of integrating factors. If the equation

$$\mu\,[M(x, y)dx + N(x, y)dy] = 0, \tag{9.2.5}$$

is exact, the integrating factor μ satisfies the relation

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N). \tag{9.2.6}$$

Or,

$$\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) + M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = 0. \tag{9.2.7}$$

Obviously, μ satisfies a first order partial differential equation. In general, it may not be easy to find the general solution of equation (9.2.7). However, in many special cases, this equation has an obvious solution which gives the required integrating factor μ.

After his discovery of the concept of the integrating factor, Euler went further and formulated classes of differential equations which admit of integrating factors of given kinds. He also showed that if there are two different integrating factors of a first order equation, then their ratio is a solution of the equation. Both Euler and Clairaut played a major role in the development of the theory and method of integrating factors.

Another major advance was made by Euler who solved by reducing a particular class of the second order ordinary differential equations to equations of the first order. Euler's method involves in replacing x and y by new variables u and v by the substitution $x = e^{ru}$ and $y = ve^u$, where r is a constant to be determined. With a suitable choice of r and a new variable $w = \frac{du}{dv}$, the original second order equation reduces to the first order in w and v. This method is fairly general in the sense that many ordinary differential equations of order higher than the second can be reduced to a lower order by similar methods. It may be appropriate to recall Euler's work on the Riccati differential equation

$$\frac{dy}{dx} + y^2 = a\,x^n. \tag{9.2.8}$$

If one particular solution u is known, then the transformation $y = u + \frac{1}{v}$ produces an ordinary linear equation. If two particular integrals are known, then the original equation can be reduced to the problem of quadrature.

In his letter to Johann Bernoulli in 1739, Euler gave a general treatment of the homogeneous ordinary linear equation of the nth order with constant coefficients in the form

$$a_n\,y^{(n)} + a_{n-1}\,y^{(n-1)} + \cdots + a_1\,y' + a_0\,y = 0, \tag{9.2.9}$$

where a_0, a_1, $\cdots$, a_n are constants. He obtained the general solution by a substitution of $y = e^{mx}$ so that m satisfies the *auxiliary equation* of nth degree

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0. \tag{9.2.10}$$

The n roots of (9.2.10) produce n particular solutions and then the general solution can be obtained by adding n particular solutions each multiplied by an arbitrary constant. He also discussed all possible cases involving real distinct, real equal, complex conjugate and multiple complex roots of (9.2.10). Thus, Euler completely solved a class of nth order homogeneous linear equations with constant coefficients. Subsequently, he also gave a method of solution of the nonhomogeneous nth order equating where zero on the right hand side of (9.2.9) is replaced by a function $f(x)$. His method was to multiply the nonhomogeneous equation by $e^{mx}dx$, integrate both sides and proceed to determine m so as to reduce the equation to one of lower order.

In his subsequent work starting from 1740, Euler generalized the method to solve the nth order *Euler* (or *equidimensional*) *equation* with variable coefficients in the form

$$a_n\,x^n y^{(n)} + a_{n-1}\,x^{n-1}y^{(n-1)} + \cdots + a_1\,xy' + a_0\,y = 0, \tag{9.2.11}$$

where a_n, a_{n-1}, $\cdots$, a_0 are real constants. This equation is often called the *Cauchy–Euler equation*. Using a trial solution $y = x^m$, where m is to be determined, it turns out that m satisfies an auxiliary equation of degree n in the form

$$\begin{aligned} F(m) \equiv a_n m(m-1)\cdots(m-n+1) + a_{n-1}\,[m(m-1)\cdots(m-n+2)] \\ + \cdots + a_1 m + a_0 = 0, \end{aligned} \tag{9.2.12}$$

where $F(m)$ is a polynomial in m of degree n. Thus, the functions x^{m_1}, x^{m_2}, $\cdots$, x^{m_n} corresponding to roots m_1, m_2, $\cdots$, m_n of $F(m) = 0$ are n particular solutions of equation (9.2.11). Hence, the general solution of (9.2.11) can be determined by adding n particular solutions each multiplied by an arbitrary constant. Euler also extended his method of solution for the nth order linear nonhomogeneous Cauchy–Euler equation (9.2.11), where zero on the right hand side is replaced by a function of x and then obtained the general solution. In other words, the nonhomogeneous nth order Euler equation of the form

$$\sum_{r=0}^{n} a_{n-r}\,x^{n-r}y^{(n-r)}(x) = f(x), \tag{9.2.13}$$

where a_n, a_{n-1}, $\cdots$, a_0 are constants and $a_n \neq 0$. The change of the independent variable $x = e^t$, $(x > 0)$ transforms (9.2.13) to the linear equation of order n with constant coefficients

$$\sum_{r=0}^{n} a_{n-r}\,[D(D-1)\cdots(D-r+1)]\,y = f(e^t), \tag{9.2.14}$$

where $D \equiv \frac{d}{dt}$. This can be solved by the usual methods including the method of undermined coefficients, variation of parameters or the Laplace transform method.

Beginning from 1753, Euler also considered a particular type of equation in the form

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0, \tag{9.2.15}$$

where $X = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ and $Y = a_4y^4 + a_3y^3 + a_2y^2 + a_1y + a_0$. He showed that its general solution has the form $F(x, y) = 0$, where $F(x, y)$ is a symmetric polynomial of degree 4 in x and y.

In particular, he solved the equation

$$\frac{dx}{\sqrt{1-x^2}} + \frac{dy}{\sqrt{1-y^2}} = 0, \tag{9.2.16}$$

to obtain a particular solution

$$\sin^{-1} x + \sin^{-1} y = c, \tag{9.2.17}$$

whose c is an arbitrary constant.

It follows from equation (9.2.16) that

$$\begin{aligned} y' &= -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \cdot \frac{\sqrt{(1-x^2)(1-y^2)} - xy}{\sqrt{(1-x^2)(1-y^2)} - xy} \\ &= -\frac{\left(\sqrt{1-y^2} - \frac{xy}{\sqrt{(1-x^2)}}\right)}{\left(\sqrt{1-x^2} - \frac{xy}{\sqrt{(1-y^2)}}\right)}. \end{aligned} \tag{9.2.18}$$

Or,

$$\left(\sqrt{1-y^2} - \frac{xy}{\sqrt{(1-x^2)}}\right) + y'\left(\sqrt{1-x^2} - \frac{xy}{\sqrt{(1-y^2)}}\right) = 0.$$

Or, equivalently,

$$\left(\sqrt{1-y^2} - \frac{x\,yy'}{\sqrt{(1-y^2)}}\right) + \left(y'\sqrt{1-x^2} - \frac{xy}{\sqrt{(1-x^2)}}\right) = 0.$$

Thus,

$$\frac{d}{dx}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) = 0. \tag{9.2.19}$$

This shows that equation (9.2.16) has also the solution

$$x\sqrt{1-y^2} + y\sqrt{1-x^2} = C, \tag{9.2.20}$$

where C is an arbitrary constant. Since the equation has only one distinct solution, two solutions (9.2.17) and (9.2.20) must be related to each other in a definite way so that this relation can be expressed by $C = f(c)$. If we set $x = \sin u$ and $y = \sin v$, then

$$u + v = c, \quad \text{and} \quad \sin u \cos v + \sin v \cos u = f(c) = f(u+v). \tag{9.2.21}$$

Putting $v = 0$, $\sin u = f(u)$ and hence, it turns out that

$$\sin u \cos v + \cos u \sin v = \sin(u+v). \tag{9.2.22}$$

This is the celebrated addition formula for the sine function.

The mathematical method of variation of parameters for the three body problem in celestial mechanics was described by Newton in his *Principia.* He first treated the motion of the Moon about the Earth and then determined the elliptic orbit. Newton included the effects of the Sun or the Moon's orbit by considering variations in the later. In the *Acta Eruditorum* of 1697, Johann Bernoulli used the method of variation of parameters to solve nonhomogeneous ordinary differential equations. In 1739, Euler solved the second order equation

$$y'' + k^2 y = f(x) \tag{9.2.23}$$

by the method of variation of parameters. This method was first used by Euler in 1748 to treat mutual perturbations of planets, Jupiter and Saturn and won a famous prize from the French Academy. Both Laplace and Lagrange developed the method of variation of parameters for a single nonhomogeneous nth order ordinary differential equation

$$L_n(y) \equiv a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x)\, y = f(x). \tag{9.2.24}$$

The reader is referred to Ince's (1953) book for a detailed analysis of the method of variation of parameters. In short, it is necessary to find a fundamental set of solutions $y_1, y_2, \cdots, y_n$ of the corresponding homogeneous equation

$$L_n(y) = 0 \tag{9.2.25}$$

so that the general solution of (9.2.25) or the complementary function $y_c(x)$ is given by

$$y_c(x) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n, \tag{9.2.26}$$

where $c_1, c_2, \cdots c_n$ are arbitrary constants.

The method of variation of parameters involves the replacement of the constants $c_1, c_2, \cdots c_n$ by functions $u_1(x), u_2(x), \cdots u_n(x)$. It is then necessary to determine the n functions $u_1(x)$, $u_2(x)$, $\cdots$ $u_n(x)$ so that

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x), \tag{9.2.27}$$

satisfies the nonhomogeneous equation

$$L_n(y) = f(x), \tag{9.2.28}$$

where the coefficients $a_n(x)$ of $y^{(n)}(x)$ is unity. It turns out that the set of n functions $u_1(x)$, $u_2(x), \cdots, u_n(x)$ satisfy n equations so that they can be determined. Thus, the solution $y_p(x)$ together with the solution (9.2.26) of the homogeneous equation (9.2.25) is the general solution of the nonhomogeneous equation (9.2.24). In other words, the general solution of (9.2.24) is

$$\begin{aligned} y(x) = y_c(x) + y_p(x) &= (c_1y_1 + c_2y_2 + \cdots + c_ny_n) \\ &\quad + (u_1y_1 + u_2y_2 + \cdots + u_ny_n)\,. \end{aligned} \tag{9.2.29}$$

After his discovery of the method of variation of parameters in 1774, Lagrange successfully applied the method to solve many problems of analytical mechanics and mathematical physics. Subsequently, Lagrange and Laplace wrote a large number of major papers and books on a wide variety of basic problems of analytical mechanics and celestial mechanics. In his masterpiece five-volume *Mécanique céleste* (*Celestial Mechanics*) published in 1799-1825, Laplace presented his own complete work as well as major theoretical and observational discoveries of Newton, Clairaut, d'Alembert, Lagrange and Euler. These volumes represent his brilliant contributions to the general principles of the equilibrium and motion of bodies with applications to the motion of the heavenly bodies, and the basic equations of motion of Jupiter and Saturn. Indeed, he described almost complete mathematical solutions posed by the Solar System. It is a delight to include here Laplace's conclusion that "nature ordered the celestial machine for an external duration, upon the same principles which prevail so admirably upon the earth, for the preservation of individuals and for the perpetuity of the species." With the improvement of the mathematical methods for solving differential equations and development of new physical principles, mathematical scientists of the nineteenth and twentieth centuries made some special efforts to obtain better results on many subjects of interest and on the n-body problem and the stability of the Solar System.

The most simplest of all nonhomogeneous differential equations of the nth order is

$$\frac{d^n y}{dx^n} = f(x). \tag{9.2.30}$$

The process of integration n times in succession leads to the following:

$$\frac{d^{n-1}y}{dx^{n-1}} = \int_{x_0}^{x} f(t)dt + c_1$$

$$\frac{d^{n-2}y}{dx^{n-2}} = \int_{x_0}^{x} dt \int_{x_0}^{x} f(t)dt + c_1(x - x_0) + c_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y = \int_{x_0}^{x} dt \int_{x_0}^{x} dt \cdots \int_{x_0}^{x} f(t)dt + c_1\frac{(x-x_0)^{n-1}}{(n-1)!} + \cdots + c_n, \quad (9.2.31)$$

where x_0 is a constant and $c_1, c_2, \cdots, c_n$ are arbitrary constants. The multiple integral in (9.2.31) can be replaced by a single integral so that the general solution of (9.2.30) is given by

$$y = \frac{1}{(n-1)!}\int_{x_0}^{x} (x-t)^{n-1} f(t)dt + c_1\frac{(x-x_0)^{n-1}}{(n-1)!} + \cdots + c_n, \quad (9.2.32)$$

where the first integral represents the particular integral

$$y_p(x) = \frac{1}{(n-1)!}\int_{x_0}^{x} (x-t)^{n-1} f(t)dt \quad (9.2.33)$$

which satisfies (9.2.30) as follows:

$$\frac{dy_p}{dx} = \frac{1}{(n-2)!}\int_{x_0}^{x} (x-t)^{n-2} f(t)dt$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^{n-1}y_p}{dx^{n-1}} = \int_{x_0}^{x} f(t)dt$$

and finally,

$$\frac{d^n y_p}{dx^n} = f(x).$$

Euler also developed a new method, known as the *Euler transform method*, to solve a second order ordinary differential equation in the form

$$(a_2x^2 + a_1x + a_0)y'' + (b_1x + b_0)y' + c_0y = 0, \quad (9.2.34)$$

where a_2, a_1, a_0, b_1, b_0 and c_0 are constants and coefficients of each derivative appearing in the equation is a polynomial of the same degree as the order of the derivative. This equation includes the Legendre, Chebyshev, and hypergeometric differential equations as special cases. In order to find the solution of (9.2.34), Euler introduced the solution $y(x)$ in 1769 as the *Euler transform* of a function $f(t)$ defined by

$$y(x) = E\{f(t)\} = \int_a^b K(x,t)f(t)dt = \int_a^b \frac{f(t)dt}{(x-t)^{\alpha+1}}, \quad (9.2.35)$$

where $K(x,t) = (x-t)^{-\alpha-1}$ is called the *Euler kernel* in the complex t-plane. Although the Euler kernel can be deduced for ordinary differential equations of the (9.2.34), the Euler transform is more appropriate than the Laplace transform for equations with only regular singular points. The kernel of the Laplace transform has an essential singularity at infinity, and the Laplace transform is more suitable for equations which have an irregular singular point at infinity.

To illustrate the use of the Euler transform, we obtain the Euler integral solution of the Legendre differential equation in the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \tag{9.2.36}$$

where n is an integer and equation (9.2.36) remains unchanged with the replacement of n by $-(n+1)$.

We introduce the Euler transform $y(x)$ of a function $f(t)$ defined by (9.2.35) so that the derivatives of y are

$$y' = -(\alpha+1)\int \frac{f(t)dt}{(x-t)^{\alpha+2}}, \quad y'' = (\alpha+1)(\alpha+2)\int \frac{f(t)dt}{(x-t)^{\alpha+3}}. \tag{9.2.37}$$

Substituting y' and y'' in (9.2.36) gives

$$\int \left[(\alpha+1)(\alpha+2)(1-x^2) + 2x(\alpha+1)(x-t) + n(n+1)(x-t)^2\right] \times \frac{f(t)dt}{(x-t)^{\alpha+3}} = 0. \tag{9.2.38}$$

Using results $x = (x-t)+t$ and $x^2 = [(x-t)+t]^2 = (x-t)^2 + 2t(x-t) + t^2$ in integral (9.2.38) and simplifying yields

$$\int \frac{Af(t)dt}{(x-t)^{\alpha+1}} + \int \frac{B(t)f(t)dt}{(x-t)^{\alpha+2}} + \int \frac{C(t)f(t)dt}{(x-t)^{\alpha+3}} = 0, \tag{9.2.39}$$

where

$$A = n(n+1) - \alpha(\alpha+1), \quad B(t) = -2t(\alpha+1)^2, \quad C(t) = (1-t^2)(\alpha+1)(\alpha+2). \tag{9.2.40}$$

We choose α to make $A = 0$, then the first integral in (9.2.39) is zero. Integrating by parts allows us to combine the third integral with the second integral in (9.2.39). This leads to a first order differential equation for $f(t)$ with only a single boundary term. With the choice of $\alpha = n$, result (9.2.39) can be reorganized to obtain

$$\int \frac{(n+2)}{(x-t)^{n+3}}(1-t^2)f(t)dt - \int \frac{2t(n+1)f(t)dt}{(x-t)^{n+2}} = 0. \tag{9.2.41}$$

Integrating the first integral in (9.2.41) by parts leads to

$$\int \frac{d}{dt}\left[\frac{(1-t^2)f(t)}{(x-t)^{n+2}}\right] dt - \int \frac{1}{(x-t)^{n+2}}\left[\frac{d}{dt}\{(1-t^2)f(t)\} - 2t(n+1)f(t)\right] dt = 0. \quad (9.2.42)$$

This result is satisfied by requiring each integral vanishes separately so that

$$\int \frac{d}{dt}\left[\frac{(1-t^2)f(t)}{(x-t)^{n+2}}\right] dt = 0 \quad (9.2.43)$$

which determines the domain of integration. Thus, $f(t)$ satisfies the first order differential equation

$$\frac{d}{dt}\left[(1-t^2)f(t)\right] - 2t\,(n+1)f(t) = 0. \quad (9.2.44)$$

The solution of this equation is given by

$$f(t) = (1-t^2)^n. \quad (9.2.45)$$

Therefore, the Euler transform solution of the Legendre equation is obtained in the form

$$y(x) = \int_{-1}^{1} \frac{(1-t^2)^n dt}{(x-t)^{n+1}}, \quad (9.2.46)$$

where the domain of integration is determined by (9.2.43), that is, by

$$\int \frac{d}{dt}\left[\frac{(1-t^2)^{n+1}}{(x-t)^{n+2}}\right] dt = 0. \quad (9.2.47)$$

In order to determine the appropriate domain of integration in (9.2.46) and (9.2.47), we note that the numerator of each integrand vanishes at $t = \pm 1$ and that the denominator vanishes at $t = x$. The point $x = 1$ is a regular singular point of the Legendre equation (9.2.36) and the integrands in (9.2.46) and (9.2.47) have simple poles at $t = 1$. So, if the range of x is confined to the region $|x| > 1$, then the domain of integration can be chosen such that $-1 \le t \le 1$ which ensures that (9.2.47) remains valid and integrands in (9.2.46) and (9.2.47) have no singularities. Thus, the solution of the Legendre equation takes the standard definite integral form

$$Q_n(x) = \frac{1}{2^{n+1}} \int_{-1}^{1} \frac{(1-t^2)^n dt}{(x-t)^{n+1}}. \quad (9.2.48)$$

This is usually known as the *Legendre function of the second kind* of order n. For $|x| > 1$, $Q_n(x)$ is linearly independent of the Legendre polynomials $P_n(x)$ given by the Rodrigues formula

$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]. \quad (9.2.49)$$

These polynomials are finite for any finite value of x. Thus, two solutions $P_n(x)$ and $Q_n(x)$ form a fundamental set for the Legendre equation (9.2.36) in the region $|x| > 1$ for any integer n. For $n = 1, 2, 3, \cdots$ the Rodrigues formula (9.2.49) can readily be used to calculate polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$, $\cdots$ and this formula can also be used to obtain the integral representation of the Legendre polynomials. In order to show this, it is convenient to change the real x to complex $z = x + iy$. We define the finite complex Legendre polynomials in the same way as for real x. In particular, we write the Rodrigues formula for complex z in the form

$$P_n(z) = \frac{1}{2^n\, n!}\frac{d^n}{dz^n}(z^2-1)^n. \tag{9.2.50}$$

We then recall the Cauchy integral formula for a complex function $f(t)$ which is analytic on and inside of a simple closed contour C, then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(t)}{(t-z)^{n+1}}\,dt, \tag{9.2.51}$$

where z is any point inside C. Using the Rodrigues formula, we obtain

$$P_n(z) = \frac{1}{2^n\, n!}\frac{d^n}{dz^n}(z^2-1)^n = \frac{1}{2\pi i\, 2^n}\int_C \frac{(t^2-1)^n dt}{(t-z)^{n+1}}. \tag{9.2.52}$$

This is known as the *Schläfli representation* of $P_n(z)$. To simplify (9.2.52), we next choose C to be a circle with center $t = z$ and radius $|\sqrt{z^2-1}|$ with $z \neq 1$. Putting $t = z + \sqrt{z^2-1}\, e^{i\theta}$ in (9.2.52) and simplifying gives

$$P_n(z) = \frac{1}{2\pi}\int_0^{2\pi}\left(z + \sqrt{z^2-1}\cos\theta\right)^n d\theta. \tag{9.2.53}$$

This is known as the *Laplace integral representation* for $P_n(z)$. It is also valid for $z = 1$ so that $P_n(1) = 1$.

We can then derive a simple integral relation between P_n and Q_n by integrating $Q_n(x)$ by parts n times and using (9.2.51). The resulting integral formula is known *Neumann's integral formula* in the form

$$Q_n(x) = \frac{1}{2}\int_{-1}^{1}\frac{P_n(t)}{x-t}\,dt, \tag{9.2.54}$$

where the denominator of the integrand does not vanish as $|x| > 1$ and $|t| \leq 1$.

In his 1769 *Institutiones Calculi Integralis*, Euler presented the hypergeometric differential equation

$$x(1-x)y'' + [c - (a+b+1)x]\,y' - aby = 0, \tag{9.2.55}$$

where a, b and c are constants and then derived the infinite series solution known as the *hypergeometric series*

$$\begin{aligned} y &= F(a,b,c;x) \\ &= 1 + \frac{a.b}{1.c}\,x + \frac{a(a+1)b(b+1)}{1.2.c(c+1)}x^2 \\ &\quad + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3.c(c+1)(c+2)}\,x^3 + \cdots . \end{aligned} \tag{9.2.56}$$

It was Johann Friedrich Plaff (1765-1825), Gauss' teacher and friend, who introduced the term *hypergeometric* to describe the differential equation and its solution as the *hypergeometric function.*

The Euler transform (9.2.35) can be used to solve (9.2.55). The method of solution is similar to that used for finding the solution of the Legendre equation (9.2.36). Making reference to that method of solution, we obtain the Euler's solution for $y(x)$ as

$$y(x) = \int_1^\infty t^{a-c}(1-t)^{c-b-1}(x-t)^{-a}\,dt \tag{9.2.57}$$

where $|x| < 1$ and $c > b > 0$.

Expanding the factor $(x-t)^{-a}$ and integrating each term in the series leads to the following series solution

$$y(x) = F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\,\frac{x^n}{n!}, \tag{9.2.58}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ with $(a)_0 = 1$ and $F(a,b,c;0) = 1$.

The following special cases follow from the hypergeometric function (9.2.54):

$$F(a,b,b;x) = (1-x)^a, \tag{9.2.59}$$

$$F(1,1,2;-x) = \frac{1}{x}\ln(1+x), \tag{9.2.60}$$

$$\lim_{b\to\infty} F\left(a,b,a;\frac{x}{b}\right) = e^x, \tag{9.2.61}$$

$$F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x^2\right) = \frac{1}{x}\sin^{-1}x, \tag{9.2.62}$$

$$F\left(n+1,-n,1;\frac{1-x}{2}\right) = P_n(x). \tag{9.2.63}$$

The standard integral representation of the hypergeometric function is

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-x)^{-a}\,dt. \tag{9.2.64}$$

Another equivalent form of the integral representation of the hypergeometric function is

$$F(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt, \quad (9.2.65)$$

where $c > b > 0$.

Euler also showed the following results

$$F(-n,b,c;z) = (1-z)^{c+n-b}F(c+n,c-b,c;z), \quad (9.2.66)$$

$$F(-n,b,c;z) = \frac{n!}{(c)_n} \int_0^1 t^{-n-1}(1-t)^{c+n-1}(1-zt)^{-b}dt. \quad (9.2.67)$$

During the seventeenth and eighteenth centuries, mathematicians made serious attempts to solve ordinary differential equations in terms of elementary functions and quadratures. When these methods failed, they solved equations by means of infinite series and by numerical methods. We conclude this section by adding Euler's numerical method. In 1768, Euler developed a simple finite difference method for the numerical solution of an ordinary differential equation

$$\frac{dy}{dx} = f(x,y), \quad (9.2.68)$$

with the given initial condition

$$y(x_0) = y_0. \quad (9.2.69)$$

With a uniform step size h between the points $x_0,\ x_1,\ x_2,\ \cdots$, Euler constructed points $x_{n+1} = x_0 + (n+1)h = x_n + h,\ n = 0,\ 1,\ 2,\ \cdots$, and then obtained the formula

$$y_{n+1} = y_n + h\,f(x_n, y_n) = y_n + h\,y_n' + O(h^2). \quad (9.2.70)$$

If $f(x,y)$ is continuous, then the sequence of Euler polygonal lines converges uniformly as $h \to 0$ to the unknown function $y(x)$ on a sufficiently small closed interval containing x_0. However, Euler's formula is not an accurate formula.

The modified Euler formula for the initial value problem (9.2.68)–(9.2.69) is given by

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{1}{2}h,\ y_n + \frac{1}{2}h\,y_n'\right)\right] + O(h^3). \quad (9.2.71)$$

The so-called improved Euler method is a simple refinement of the Euler method that takes into account of the average value of the gradient at the end points (x_n, y_n) and (x_{n+1}, y_{n+1}). So, the improved Euler formula for the initial value problem (9.2.68)–(9.2.69) is given by

$$\begin{aligned} y_{n+1} &= y_n + h\left[\frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{h}\right] \\ &= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_n + h, y_n + hy'n)\right] + O(h^3), \end{aligned} \tag{9.2.72}$$

where x_{n+1} has been replaced by $(x_n + h)$.

Subsequently, further refinement of the Euler method and its modifications was made by Carl Runge (1856-1927) in 1895 and M. W. Kuttta (1867-1944) in 1901 who based their work on the Taylor series approximation. The Runge-Kutta formula involves a weighted average value of $f(x, y)$ taken at different points in $x_n \leq x \leq x_{n+1}$ and it is given by

$$y_{n+1} = y_n + \frac{h}{6}\left(k_{1n} + 2k_{2n} + 2k_{3n} + k_{4n}\right) + O(h^5), \tag{9.2.73}$$

where $k_{1n} = f(x_n, y_n)$, $k_{2n} = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h\, k_{1n}\right)$, $k_{3n} = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h\, k_{2n}\right)$ and $k_{4n} = f\left(x_n + h, y_n + h\, k_{3n}\right)$. The sum $(k_{1n} + 2k_{2n} + 2k_{3n} + k_{4n})/6$ can be interpreted as an average value of the slope. The Runge-Kutta formula is one of the most accurate and successful of all one-step formulas, and hence, it has been widely used in solving initial value problems.

The Euler method has been refined by means of various modifications over the last 300 years. The major problem that must be investigated for every numerical method is the convergence of the approximate solution to the exact solution as $h \to 0$. With the advent of high speed electronic computers, the use of numerical methods to solve initial value problems has become very common. In addition to the question of convergence, there is also the question of calculating the error made in computing the values y_1, y_2, $\cdots$, y_n. Usually, this error arises from two sources: first, the formula used in the numerical method is only an approximate one that causes *truncation error*, or *discretization error*; second, any modern computer introduces *rounding errors* during computation.

Euler's method and its extensive modifications can be used to solve the more general case of solving a system of n ordinary differential equations

$$y'_n = f_n(x, y_1, y_2, \cdots, y_n), \qquad n = 1, 2, \cdots, n \tag{9.2.74}$$

with the given initial conditions

$$y_n(x_0) = y_{n_0}. \tag{9.2.75}$$

The new general methods for solving ordinary differential equations were yet to be discovered. But no major new methods beyond those already discussed above were invented for a hundred years or so, until the Laplace transform and operator methods were introduced at the end of the nineteenth century. It must be recognized that Euler's analytical and numerical methods have served as the fundamental basis for all subsequent developments in the classical and modern theories and computations of ordinary differential equations.

9.3 Euler's Work on Partial Differential Equations

During the middle of the eighteenth century, the wave equation and its methods of solution attracted the attention of many celebrated mathematicians including Johann Bernoulli, Leonhard Euler, Daniel Bernoulli, J. L. Lagrange, and Jean d'Alembert. It was d'Alembert who first derived the one-dimensional wave equation for vibration of an elastic string and solved it in 1746. Some form of this equation or its various generalizations almost frequently arose in any mathematical analysis of propagation of waves in continuous media. In fact, the studies of water waves, acoustic waves, elastic waves in solids, and electromagnetic waves are all based on this equation. A classical technique known as the *method of separation of variables* is perhaps one of the oldest systematic methods for solving partial differential equations including the wave equation, the diffusion equation and the Laplace equation (or potential equation).

Euler considered a more general order partial differential equation, known as the *Euler equation*, in the form

$$a\,\frac{\partial^2 u}{\partial x^2} + 2h\,\frac{\partial^2 u}{\partial x \partial y} + b\,\frac{\partial^2 u}{\partial y^2} = 0, \tag{9.3.1}$$

where a, h and b are constants.

We first define two new independent variables ξ and η by the linear relations

$$\xi = px + qy, \qquad \eta = rx + sy, \tag{9.3.2}$$

where p, q, r and s are arbitrary constants. Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = p\,\frac{\partial u}{\partial \xi} + r\,\frac{\partial u}{\partial \eta}, \tag{9.3.3}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = q\,\frac{\partial u}{\partial \xi} + s\,\frac{\partial u}{\partial \eta}, \tag{9.3.4}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(p\,\frac{\partial}{\partial \xi} + r\,\frac{\partial}{\partial \eta}\right)\left(p\,\frac{\partial u}{\partial \xi} + r\,\frac{\partial u}{\partial \eta}\right)\\ &= p^2\,\frac{\partial^2 u}{\partial \xi^2} + 2pr\,\frac{\partial^2 u}{\partial \xi \partial \eta} + r^2\,\frac{\partial^2 u}{\partial \eta^2},\end{aligned} \tag{9.3.5}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = q^2\,\frac{\partial^2 u}{\partial \xi^2} + 2sq\,\frac{\partial^2 u}{\partial \xi \partial \eta} + s^2\,\frac{\partial^2 u}{\partial \eta^2}, \tag{9.3.6}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \left(p\,\frac{\partial}{\partial \xi} + r\,\frac{\partial}{\partial \eta}\right)\left(q\,\frac{\partial u}{\partial \xi} + s\frac{\partial u}{\partial \eta}\right)\\ &= pq\,\frac{\partial^2 u}{\partial \xi^2} + (rq + sp)\,\frac{\partial^2 u}{\partial \xi \partial \eta} + rs\,\frac{\partial^2 u}{\partial \eta^2}.\end{aligned} \tag{9.3.7}$$

Substituting these results for the second partial derivatives into (9.3.1) gives

$$\begin{aligned}\left(ap^2 + 2hpq + bq^2\right)\frac{\partial^2 u}{\partial \xi^2} &+ 2\left[apr + bsq + h(rq + sp)\right]\frac{\partial^2 u}{\partial \xi \partial \eta}\\ &+ \left(ar^2 + 2hrs + bs^2\right)\frac{\partial^2 u}{\partial \eta^2} = 0.\end{aligned} \tag{9.3.8}$$

We now choose p, q, r and s such that $p = r = 1$ and such that q and s are the two roots λ_1 and λ_2 of the quadratic equation

$$a + 2h\lambda + b\lambda^2 = 0 \tag{9.3.9}$$

so that

$$\lambda_1 + \lambda_2 = -\frac{2h}{b} \quad \text{and} \quad \lambda_1\lambda_2 = \frac{a}{b}. \tag{9.3.10}$$

Consequently, equation (9.3.8) reduces to

$$\left[a + h\left(\lambda_1 + \lambda_2\right) + b\lambda_1\lambda_2\right]\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{9.3.11}$$

which is, by (9.3.10),

$$\frac{2}{b}(ab - h^2)\frac{\partial^2 u}{\partial \xi \partial \eta} = 0. \tag{9.3.12}$$

We characterize the Euler equation (9.3.1) as *hyperbolic* (roots λ_1, λ_2 are real and distinct), *parabolic* (roots λ_1, λ_2 are real and equal), or *elliptic* (roots λ_1, λ_2 are complex). If equation (9.3.1) is not parabolic ($ab - h^2 \neq 0$), equation (9.3.12) may be integrated to give the general solution

$$u(x, y) = \phi(\xi) + \psi(\eta) = \phi(x + \lambda_1 y) + \psi(x + \lambda_2 y). \tag{9.3.13}$$

where ϕ and ψ are arbitrary functions, λ_1 and λ_2 are the roots of (9.3.9). This is the case for the hyperbolic and elliptic equations depending on $h^2 - ab > 0$ or < 0. In the former case, the general solution (9.3.13) is the sum of two arbitrary functions of real arguments. In the latter case, the roots λ_1, λ_2 of (9.3.9) are complex conjugates with $\alpha \pm i\beta$ so that $\xi = x(\alpha + i\beta)y$ and $\eta = x + (\alpha - i\beta)y = \xi^*$. Consequently, equation (9.3.12) becomes

$$\frac{\partial^2 u}{\partial \xi \partial \xi^*} = 0 \tag{9.3.14}$$

and the general solution is

$$u(x, y) = \phi(\xi) + \psi(\xi^*) = \phi(x + \alpha y + i\beta y) + \psi(x + \alpha y - i\beta y). \tag{9.3.15}$$

Thus, the general solution (9.3.15) is the sum of two arbitrary functions of complex arguments which is the general property of the solution of elliptic equations.

For the parabolic case, $ab - h^2 = 0$ and (9.3.12) is satisfied identically. The roots λ_1 and λ_2 are equal. We choose $p = 1$ and keep q, r and s are arbitrary. Then

$$(a+2hq+bq^2)\frac{\partial^2 u}{\partial \xi^2} + 2[ar+bsq+h(rq+s)]\frac{\partial^2 u}{\partial \xi \partial \eta} + (ar^2+2hrs+bs^2)\frac{\partial^2 u}{\partial \eta^2} = 0. \tag{9.3.16}$$

If q is now chosen to be root of $a + 2hq + bq^2 = 0$, then $q = -\frac{h}{b}$, is the double root because $ab - h^2 = 0$. Consequently, the first term of (9.3.16) is zero and the middle term of (9.3.16) is also zero because $ab - h^2 = 0$.

Thus, if r and s are not both zero, equation (9.3.16) becomes

$$\frac{\partial^2 u}{\partial \eta^2} = 0. \tag{9.3.17}$$

Integrating this equation gives the general solution

$$u(x, y) = \phi(\xi) + \eta\,\psi(\xi), \tag{9.3.18}$$

where ϕ and ψ are arbitrary functions and since $p = 1$, $q = -h/b = \lambda$, $\xi = x + \lambda y$ and $\eta = rx + sy$ (r, s are arbitrary, but not both zero). Hence, the general solution (9.3.18) reduces to

$$u(x, y) = \phi(x + \lambda y) + (rx + sy)\,\psi(x + \lambda y). \tag{9.3.19}$$

The above solution can be applied to the partial differential equation of a stretched elastic string of constant line density ρ and constant tension T^* or plane waves of sound in the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2} \tag{9.3.20}$$

where $c^2 = T^*/\rho$.

Using $\xi = x + \lambda_1 t$ and $\eta = x + \lambda_2 t$, where λ_1 and λ_2 are the roots of $\lambda^2 - c^2 = 0$, that $\lambda_1 = c$ and $\lambda_2 = -c$. Consequently, the general solution of (9.3.20) is

$$u(x, y) = \phi(x + ct) + \psi(x - ct). \tag{9.3.21}$$

This solution represents the sum of two waves, the first one traveling to the left with constant velocity c and the second one traveling to the right with the same velocity c.

We next obtain the d'Alembert solution of the one-dimensional wave equation (9.3.20) subject to the initial conditions

$$u(x, 0) = f(x) \quad \text{and} \quad \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x), \tag{9.3.22}$$

where $f(x)$ and $g(x)$ represent the initial displacement and the initial velocity at time $t = 0$.

The general solution of (9.3.20) is given by (9.3.21) in the form

$$u(x, t) = \phi(x + ct) + \psi(x - ct), \tag{9.3.23}$$

where ϕ and ψ are arbitrary twice differentiable functions of real arguments.

Using the initial conditions (9.3.22) gives

$$\phi(x) + \psi(x) = f(x), \qquad c\,\phi'(x) - c\,\psi'(x) = g(x), \tag{9.3.24}$$

where the prime denote the derivative. Integrating the second equation, we have

$$\phi(x) - \psi(x) = \frac{1}{c}\int_a^x g(\xi)d\xi, \tag{9.3.25}$$

where the constant of integration has been included in the lower limit by introducing an arbitrary constant a. From the first equation of (9.3.24) and (9.3.25), we obtain

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_a^x g(\xi)d\xi, \tag{9.3.26}$$

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_a^x g(\xi)d\xi. \tag{9.3.27}$$

Consequently, the solution (9.3.23) can be expressed in the form

$$u(x, t) = \frac{1}{2}\left[f(x + ct) + f(x - ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi. \tag{9.3.28}$$

This is the celebrated *d'Alembert solution*. It is worth noting that the value of the displacement function $u(x, t)$ depends only on the initial values at points between $x - ct$ and $x + ct$ and *not* at all on initial values outside the interval $[x - ct, x + ct]$. This interval is called the *domain of dependence* of variable (x, t). Clearly, the solution (9.3.28) is unique and stable as it depends continuously on the values of the initial conditions (9.3.22).

In particular, if the string is displaced with the initial displacement $f(x)$ and zero initial velocity so that $g(x) = 0$, then the solution (9.3.28) becomes

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]. \tag{9.3.29}$$

This represents two like waves and each one has half the original displacement.

On the other hand, if the string is displaced from zero displacement with arbitrary velocity so that $f(x) = 0$ and $g(x) \neq 0$. With $g(x) = \frac{dh}{dx}$, the solution (9.3.28) assumes the form

$$u(x,t) = \frac{1}{2c}\left[h(x+ct) - h(x-ct)\right]. \tag{9.3.30}$$

Thus, the two waves are opposite to each other.

We next include here Euler's basic paper of 1749 dealing with all possible motions (periodic in both x and t) of a vibrating sting of finite length ℓ. In this case, the displacement function $u(x,t)$ satisfies the wave equation (9.3.20) in $0 < x < \ell$ and $t > 0$ subject to the same initial conditions (9.3.22) and boundary conditions $u(0,t) = 0 = u(\ell,t)$ for $t > 0$. Using the separable solution $u(x,t) = X(x)T(t) \neq 0$, the problem reduces to the eigenvalue problem (see Myint-U and Debnath (2008)) with an infinite set of eigenvalues $\lambda_n = -\left(\frac{n\pi}{\ell}\right)^2$, and the corresponding eigenfunctions $\sin\left(\frac{n\pi x}{\ell}\right)$, where $n = 1, 2, 3, \cdots$. Thus, the nth normal modes of vibration or nth harmonic can be represented by

$$u_n(x,t) = \left(a_n \cos\frac{n\pi ct}{\ell} + n_n \sin\frac{n\pi ct}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right), \tag{9.3.31}$$

$$= \frac{1}{2}a_n\left[\sin\frac{n\pi}{\ell}(x-ct) + \sin\frac{n\pi}{\ell}(x+ct)\right]$$
$$+\frac{1}{2}b_n\left[\cos\frac{n\pi}{\ell}(x-ct) + \cos\frac{n\pi}{\ell}(x+ct)\right], \tag{9.3.32}$$

where a_n and b_n are constants to be determined by the initial conditions (9.3.22). Thus, the modes of vibration consists of terms

$$\sin\frac{n\pi}{\ell}(x \pm ct) \quad \text{and} \quad \cos\frac{n\pi}{\ell}(x \pm ct), \tag{9.3.33}$$

which are periodic not only in t, but in x so that the *wavelength* of vibration is given by $\left(\frac{n\pi\lambda}{\ell}\right) = 2\pi$ or $\lambda = \lambda_n = \frac{2\ell}{n}$, where $n = 1, 2, 3, \cdots$.

On the other hand, if we denote the discrete spectrum of *circular* (or *radian*) frequencies are defined by ω_n, then $\omega_n = \left(\frac{n\pi c}{\ell}\right)$ so that the *angular frequencies* ν_n are given by $\nu_n = (\omega_n/2\pi) = \left(\frac{nc}{2\ell}\right)$, $n = 1, 2, 3, \cdots$. Thus, the periodic times, $T_n = \frac{2\pi}{\omega_n} = \left(\frac{2\ell}{nc}\right)$. When $n = 1$, $\omega_1 = \left(\frac{\pi c}{\ell}\right) = \frac{1}{\ell}\sqrt{\frac{T^*}{\rho}}$ is called the *fundamental circular frequency* and all other harmonics for $n > 1$ are known as *fundamental circular frequency.* Thus, the fundamental angular frequency is

$$\nu_1 = \frac{1}{2\ell}\sqrt{\frac{T^*}{\rho}}. \tag{9.3.34}$$

This is usually called the *fundamental* (or *Mersenne*) *law* of a stringed musical instrument. Evidently, the angular frequency of the fundamental mode of transverse vibration of a string varies as the square root of the tension, T^*, inversely as length, ℓ, and inversely as the square root of the density. The period of the fundamental mode is $T_1 = \left(\frac{2\ell}{c}\right)$ so that $T_n = \frac{1}{n} \cdot T_1$, where T_1 is often called the *fundamental period*. Clearly, $\omega_n = n\omega_1$ (or $\nu_n = n \cdot \nu_1$) which are all integral multiple of the fundamental frequency. This is the main reason why the stringed musical instruments produce sweeter musical sound (or tones) than drum instruments. Furthermore, the wavelength of vibration $\lambda_n = \frac{2\ell}{n} = c \cdot T_n$ which is equal to the distance traveled by the wave traveling with speed c during the time T_n.

Since the wave equation (7.3.20) is linear and homogeneous, the most general solution is the superposition of an infinite number of harmonics in the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi ct}{\ell} + b_n \sin\frac{n\pi ct}{\ell}\right) \sin\frac{n\pi x}{\ell}, \tag{9.3.35}$$

provided the series converges and is twice continuously differentiable with respect to x and t.

Applying the initial conditions (9.3.22)

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right), \tag{9.3.36}$$

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi c}{\ell}\right) \sin\left(\frac{n\pi x}{\ell}\right), \tag{9.3.37}$$

whence the coefficients a_n and b_n are given by

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right) dx, \quad b_n = \left(\frac{2}{n\pi c}\right) \int_0^{\ell} g(x) \sin\left(\frac{n\pi x}{\ell}\right) dx. \tag{9.3.38}$$

The nth normal modes of vibration is given by (9.3.31) which has nodes at the points

$$x = 0,\ \frac{\ell}{n},\ \frac{2\ell}{n},\ \cdots\ \frac{n\ell}{\ell}, \tag{9.3.39}$$

and so the nth modes divides the string into n equal segments of length $\left(\frac{\ell}{n}\right)$.

It is important to note that Euler did not specify whether the series in (9.3.35) contains a finite or infinite number of terms. However, Euler has the remarkable idea of superposition of vibrating modes. Thus, Euler's major point of disagreement with d'Alembert is that his solution admits all kinds of initial curves and hence, non-analytic solutions, whereas only analytic initial curves and solutions were acceptable to d'Alembert. In introducing his *discontinuous* functions, Euler wrote to d'Alembert on December 20, 1763 that "considering such functions as are subject to no law of continuity [analyticity] opens to us a wholly new range of analysis". Although Euler and d'Alembert agreed on the form of their equations, they had very different ideas as to what functions would qualify as initial data and so as solutions of the wave equation. The controversy continued between them for many years. In the mean time, Daniel Bernoulli and Lagrange made an unacceptable attempt to resolve the controversy.

While the controversy over the problem of the vibrating string continued, the major interest in the extensions of the wave equation prompted further research in problems of wave propagation. In 1762, Euler considered the problem of vibrating string with variable thickness in his paper "On the Vibratory Motion of Non-Uniformly Thick Strings". He declared that the general solution is almost beyond the power of mathematical analysis, and so, he obtained the solution of the problem for a particular case where the mass distribution m is given by

$$m = \frac{\rho}{\left(1 + \frac{x}{a}\right)^4}, \tag{9.3.40}$$

where ρ and a are constants. Then it follows that the solution is given by

$$u(x,t) = \frac{1}{A}\left[\phi(Ax + c_0 t) + \psi(Ax - c_0 t)\right], \tag{9.3.41}$$

where $\frac{1}{A} = \left(1 + \frac{x}{a}\right)$, $c_0 = \sqrt{T/\rho}$ and T is the constant tension of the string. The frequencies of the modes of vibration (or harmonics) are

$$\omega_n = \left(\frac{n}{2\ell}\right)\left(1 + \frac{\ell}{a}\right)c_0, \qquad n = 1, 2, 3, \cdots. \tag{9.3.42}$$

Therefore, the ratio of two successive frequencies is the same for a string of uniform thickness, but the fundamental frequency ω_1 is no longer inversely proportional to the length of the string.

In his paper of 1762, Euler also investigated the vibration of a string of two lengths, a and b of different thickness r and s, and obtained the frequency equation for the modes of vibration in the form

$$m \tan\left(\frac{\omega a}{m}\right) + n \tan\left(\frac{\omega b}{m}\right) = 0, \tag{9.3.43}$$

where ω is the frequency which was determined in special cases. The solutions of (9.3.12) are referred to as the *characteristic values* (or *eigenvalues*) of the string problem. It is also evident from (9.3.12) that the characteristic frequencies are *not* integral multiples of the fundamental frequency.

On the other hand, d'Alembert also considered the problems of vibration of string with constant and variable thickness in 1763 and used the method of separation of variables introduced by him. In the same paper, he wrote the wave equation in the form

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \ell, \quad t > 0, \tag{9.3.44}$$

$$u(0,t) = 0 = u(\ell,t), \quad t > 0. \tag{9.3.45}$$

He assumed the solution of the form

$$u(x,t) = \eta(x)\cos(\lambda\pi t), \tag{9.3.46}$$

so that (9.3.44) reduces to equation for $\eta(x)$ as

$$\frac{\partial^2 \eta}{\partial x^2} = -\frac{\lambda^2\pi^2}{c^2(x)}\,\eta, \qquad \eta(0) = 0 = \eta(\ell). \tag{9.3.47}$$

This is called the *boundary value* or (*eigenvalue*) *problem* for ordinary differential equations. He showed that there are infinitely many values of λ.

In 1781, Euler also investigated the transverse vibration of a heavy continuous horizontal string in his paper entitled 'On the modifying effect of their own weight on the motion of strings'. He solved the associated wave equation in the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2} + g, \qquad 0 < x < \ell, \quad t > 0, \tag{9.3.48}$$

where c is a constant and the boundary conditions are $u(0,t) = 0 = u(\ell,t)$ for $t > 0$. His solution is

$$u(x,t) = -\frac{1}{2c^2}\,g\,x(x-\ell) + \phi(x+ct) + \psi(x-ct). \tag{9.3.49}$$

This solutions are in agreement with that of the vibration of string of zero weight, where $g = 0$, except that the oscillation takes place about the parabolic curve of equilibrium

$$u = -\left(\frac{g}{2c^2}\right) x(x-\ell). \tag{9.3.50}$$

In 1759, Euler investigated the vibration of a rectangular membrane of dimensions a and b so that $0 \le x \le a$ and $0 \le x \le b$ and derived the equation of motion for the vertical displacement $u(x, y, t)$ of the membrane as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{9.3.51}$$

where c is a constant determined by the mass and tension. Euler solved (9.3.20) by assuming separable solution in the form

$$u(x, y, t) = v(x, y)\sin(\omega t + \varepsilon), \tag{9.3.52}$$

ω is the frequency of vibration and ε is the phase so that $v(x, y)$ satisfies the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\omega^2}{c^2}\, v = 0. \tag{9.3.53}$$

This equation admits the sinusoidal solutions of the form

$$v(x, y) = \sin\left(\frac{\alpha x}{a} + \varepsilon_1\right)\sin\left(\frac{\beta y}{b} + \varepsilon_2\right), \tag{9.3.54}$$

where

$$\frac{\omega^2}{c^2} = \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2}\right). \tag{9.3.55}$$

With zero initial velocity, ε_1 and ε_2 may be set zero. For fixed boundaries of the membrane, we find $\alpha = m\pi$ and $\beta = n\pi$, where m and n are integers so that the frequency $\omega = \omega_{mn}$ is given by

$$\omega_{mn} = \pi c \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{\frac{1}{2}}. \tag{9.3.56}$$

For the vibration of a circular membrane of radius $r = a$, Euler wrote equation (9.3.20) for $u(r, \theta, t)$ in polar coordinates (r, θ) in the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right). \tag{9.3.57}$$

He sought solutions of (9.3.26) in the form

$$u(r, \theta, t) = v(r)\sin(\omega t + \varepsilon_1)\sin(\nu\theta + \varepsilon_2), \tag{9.3.58}$$

where $v(r)$ satisfies the Bessel equation

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} + \left(\frac{\omega^2}{c^2} - \frac{\nu^2}{r^2}\right) v = 0. \tag{9.3.59}$$

Euler used an infinite series solution of (9.3.59) and obtained

$$v\left(\frac{\omega r}{c}\right) = r^{\nu}\left[1 - \frac{1}{1.(\nu+1)}\left(\frac{\omega r}{2c}\right)^2 + \frac{1}{1.2(\nu+1)(\nu+2)}\left(\frac{\omega r}{2c}\right)^4 + \cdots\right], \tag{9.3.60}$$

which can be written in terms of the Bessel function $J_{\nu}\left(\frac{\omega r}{c}\right)$ as

$$v\left(\frac{\omega r}{c}\right) = \left(\frac{c}{\omega}\right)^{\nu} 2^{\nu}\,\Gamma(\nu+1)\,J_{\nu}\left(\frac{\omega r}{c}\right). \tag{9.3.61}$$

For a fixed edge $r = a$, it turns out that

$$J_{\nu}\left(\frac{\omega a}{c}\right) = 0. \tag{9.3.62}$$

Since $u(r,\theta,t)$ is periodic of period 2π in θ, ν must be integer so that for a fixed ν, there are infinitely many roots ω. Euler was unsuccessful in finding a second solution of (9.3.59). Subsequently, S. D. Poisson fully developed the theory of vibrating elastic membrane.

In his 1759 paper ‘On the Propagation of Sound’, Euler considered the propagation of sound in one space dimension as waves of small amplitude and derived the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}, \qquad c^2 = 2gh, \tag{9.3.63}$$

where $u(x,t)$ is the amplitude of the wave, g is the gravitational acceleration and h is a constant relating the pressure and density.

Euler then extended his work to the two-dimensional wave propagation equation in the form

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y}\right), \quad \frac{\partial^2 v}{\partial t^2} = c^2\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y}\right), \tag{9.3.64ab}$$

where u and v are the wave amplitudes in the x and y directions respectively. He obtained the plane wave solutions of the form

$$u = k\phi\,(kx + \ell y + c\kappa t)\,, \qquad v = \ell\phi\,(kx + \ell y + c\kappa t)\,, \tag{9.3.65ab}$$

where k and ℓ are arbitrary constants, $\kappa = (k^2+\ell^2)^{\frac{1}{2}}$ and ϕ is an arbitrary function. He used the divergence of the displacement $w = u_x + v_y$ so that w satisfies the two-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right). \tag{9.3.66}$$

He recognized the need for the superposition of solutions in order to determine the general solution subject to some initial data, that the given value of w or of u and v at $t = 0$.

Introducing $z = \sqrt{x^2 + y^2}$ and $w = f(z, t)$, $u = xw$ and $v = yw$, Euler obtained the partial differential equation from (9.3.66) as

$$\frac{\partial^2 w}{\partial t^2} = c^2 \left(\frac{3}{z} \frac{\partial w}{\partial z} + \frac{\partial^2 w}{\partial z^2} \right). \tag{9.3.67}$$

Similarly, he derived the three-dimensional wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right). \tag{9.3.68}$$

Using the above equations, Euler investigated both cylindrical and spherical waves. The fundamental equation for the spherical waves is

$$\frac{\partial^2 W}{\partial t^2} = c^2 \left(\frac{\partial^2 W}{\partial r^2} + \frac{4}{r} \frac{\partial W}{\partial r} \right) \tag{9.3.69}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

Both Euler and Lagrange independently did considerable work on cylindrical and spherical waves. On the other hand, the propagation of sound waves in air was studied extensively by Daniel Bernoulli, Euler and Lagrange. They published numerous papers on this subject with special attention to the fundamental harmonic and overtones produced by a wide variety of musical instruments.

A more general form of the wave equation is

$$u_{tt} - c^2(x) u_{xx} = 0, \tag{9.3.70}$$

where c is a function of x only. The characteristic coordinates are now given by

$$\xi = t - \int^x \frac{d\tau}{c(\tau)}, \quad \eta = t + \int^x \frac{d\tau}{c(\tau)}. \tag{9.3.71ab}$$

Thus,

$$\begin{aligned} u_x &= -\frac{1}{c} u_\xi + \frac{1}{c} u_\eta, \quad u_t = u_\xi + u_\eta, \\ u_{xx} &= \frac{1}{c^2} (u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) - (u_\eta - u_\xi) \frac{c'(x)}{c^2}, \\ u_{tt} &= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}. \end{aligned}$$

Consequently, equation (9.3.70) reduces to

$$4u_{\xi\eta} + c'(x)(u_\eta - u_\xi) = 0. \tag{9.3.72}$$

In order to express c' in terms of ξ and η, we observe that

$$\eta - \xi = 2\int^x \frac{d\tau}{c(\tau)}, \tag{9.3.73}$$

so that x is a function of $(\eta - \xi)$. Thus, $a'(x)$ will be some function of $(\eta - \xi)$.

In particular, if $c(x) = A\,x^n$, where A is a constant, so that $c'(x) = n\,Ax^{n-1}$, and when $n \neq 1$, result (9.3.73) gives

$$\eta - \xi = -\frac{2}{A}\frac{1}{(n-1)}\frac{1}{x^{n-1}} \tag{9.3.74}$$

so that

$$c'(x) = -\frac{2n}{(n-1)} \cdot \frac{1}{\eta - \xi}.$$

Thus, equation (9.3.72) reduces to the form

$$4u_{\xi\eta} - \frac{2n}{(n-1)}\frac{1}{(\eta-\xi)}\left(u_\eta - u_\xi\right) = 0.$$

Finally, we find that

$$u_{\xi\eta} = \frac{n}{2\,(n-1)}\frac{1}{(\eta-\xi)}\left(u_\eta - u_\xi\right). \tag{9.3.75}$$

When $n = 1$, $c(x) = Ax$, and $c'(x) = A$, substituting $\xi = \frac{\alpha}{A}$ and $\eta = \frac{\beta}{A}$ can be used to reduce equation (9.3.72) to

$$u_{\alpha\beta} = \frac{1}{4}\left(u_\alpha - u_\beta\right). \tag{9.3.76}$$

Equation (9.3.75) is called the *Euler–Darboux equation* which has the hyperbolic form

$$u_{xy} = \frac{m}{x-y}\left(u_x - u_y\right), \tag{9.3.77}$$

where m is a positive integer.

We next note that

$$\frac{\partial^2}{\partial x \partial y}\left[(x-y)u\right] = \frac{\partial}{\partial x}\left[(x-y)\frac{\partial u}{\partial y} - u\right] = (x-y)u_{xy} + \left(u_y - u_x\right). \tag{9.3.78}$$

When $m = 1$, equation (9.3.77) becomes

$$(x-y)u_{xy} = u_x - u_y$$

so that (9.3.78) reduces to

$$\frac{\partial^2}{\partial x \partial y}\left[(x-y)u\right] = 0. \tag{9.3.79}$$

This shows that the solution of (9.3.79) is $(x-y)u = \phi(x) + \psi(y)$. Hence, the solution of (9.3.77) with $m = 1$ is

$$u(x, y) = \frac{\phi(x) + \psi(y)}{x - y}, \tag{9.3.80}$$

where ϕ and ψ are arbitrary functions.

We multiply (9.3.77) by $(x - y)$, and apply the derivative $\frac{\partial^2}{\partial x \partial y}$, so that the result is, due to (9.3.78),

$$(x-y)\frac{\partial^2}{\partial x \partial y}(u_{xy}) + \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)(u_{xy}) = m\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)u_{xy}.$$

Or,

$$(x-y)\frac{\partial^2}{\partial x \partial y}(u_{xy}) = (m+1)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(u_{xy}). \tag{9.3.81}$$

Hence, if u is a solution of (9.3.77), then u_{xy} is a solution of (9.3.77) with m replaced by $m+1$. When $m = 1$, the solution is given by (9.3.80), and hence, the solution of (9.3.77) takes the form

$$u(x, y) = \frac{\partial^{2m-2}}{\partial x^{m-1} \partial y^{m-1}}\left[\frac{\phi(x) + \psi(y)}{x - y}\right], \tag{9.3.82}$$

where ϕ and ψ are arbitrary functions.

In deriving the wave equation (9.3.20) of motion of a string of constant tension T^*. If T^* is a function of x, then the wave equation (9.3.20) must be of the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(T^*(x)\frac{\partial u}{\partial x}\right). \tag{9.3.83}$$

Introducing the normal displacement function $u(x,t) = y(x)\cos\omega t$ equation (9.3.83) assumes the form

$$\frac{d}{dx}\left[T^*(x)\frac{dy}{dx}\right] + \lambda y = 0, \qquad \lambda = \rho\omega^2. \tag{9.3.84}$$

For a heavy chain of length ℓ hanging vertically from one end, $T^* = g\rho x$, where x is measured from the free end so that (9.3.84) becomes

$$\frac{d}{dx}\left(x\,\frac{dy}{dx}\right) + \lambda y = 0, \qquad \lambda = (\omega^2/g). \tag{9.3.85}$$

In 1733, Daniel Bernoulli derived this second order equation for the displacement function. Introducing the variable $x = z^2$ so that

$$2z\frac{dz}{dx} = 1, \qquad \frac{dy}{dx} = \frac{dz}{dx}\cdot\frac{dy}{dz} = \frac{1}{2z}\,\frac{dy}{dz}, \tag{9.3.86}$$

$$\frac{d}{dx}\left(x\,\frac{dy}{dx}\right) = \frac{1}{2z}\,\frac{d}{dz}\left(\frac{z^2}{2z}\cdot\frac{dy}{dz}\right) = \frac{1}{4z}\,\frac{d}{dz}\left(z\,\frac{dy}{dz}\right), \tag{9.3.87}$$

equation (9.3.85) reduces to the form

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + c^2 y = 0, \qquad c^2 = 4\lambda, \tag{9.3.88}$$

which is, by a change of variable, $cz = \xi$,

$$\frac{d^2y}{d\xi^2} + \frac{1}{\xi}\frac{dy}{d\xi} + y = 0. \tag{9.3.89}$$

This is the classical Bessel equation which admits an infinite series solution in terms of the Bessel function of the order zero in the form

$$y = A\,J_0(\xi) = A\,J_0\left(2\sqrt{\lambda x}\right), \tag{9.3.90}$$

A is a constant. If $x = \ell$ is a node, then $J_0\left(2\sqrt{\lambda\ell}\right) = 0$ which is an eigenvalue equation for eigenvalues λ. This equation has infinitely many roots. For each λ, there is mode of oscillation and a characteristic frequency.

9.4 Euler and the Calculus of Variations

In order to bargain land with Libyans for the expansion his own great city of Carthage, the Queen Dido needed to find out what shape of a plane curve enclose a maximum area. The natural generalization of Dido's famous problem led in time to the formulation of the isoperimetric problem of the calculus of variations.

At the end of the seventeenth century, many fundamental questions and problems in geometry and mechanics deal with minimizing or maximizing of certain integrals for two major reasons. The first of these were several existence problems, such as, the Fermat principle of least time (that is, a ray of light travels in a homogeneous medium from one point to another along a path in a minimum time), brachistos. Newton's problem of missile of least resistance, Bernoulli's isoperimetric problem, Bernoulli's problem of brachistochrone (*brachistos* means shortest, *chronos* means time), and the problem of finding the surface of minimum area bounded by a closed curve in space due to Joseph Plateau, the Belgian physicist. The second reason was somewhat philosophical due to Euler, that is, how to discover a minimizing principle in nature. The 1744 famous statement of Euler is the characteristic of the philosophical origin of what is known as the *Principle of Least Action*: "For since the fabric of the Universe is most perfect and the work of a most wise Creator, nothing at all takes place in the Universe in which some rule of maximum or minimum does not appear." The celebrated minimum problem associated with Plateau's name is now involved with partial differential equation. Plateau's brilliant experimental work on soap bubbles and liquid films provided a remarkable relationship between mathematics and experimental research. The spherical soap bubbles reveals that among all closed surfaces including a given volume, the sphere has the minimum area.

In the middle of the eighteenth century, Pierre de Maupertuis, a French mathematician and astronomer enunciated the fundamental principle, known as the *Principle of Least Action*, as a guide to the nature of the universe. He believed that this simple and grand principle would embrace all phenomena of nature and once said: "God's intention to regulate physical phenomena by a general principle of highest perfection." The principle of least action, where the action is defined as the mean value of the difference between the kinetic and potential energies of a physical system averaged over some fixed time interval. The formulation of the equations of dynamics based on this principle may be considered as one of the most remarkable discovery of mathematical sciences. Historically, Maupertuis formulated the principle of least action in an attempt to extend the fundamental principle of Pierre de Fermat in optics that, in a optically homogeneous medium, a ray of light travels from one point to another along the shortest path and in the minimum time.

Joseph Lagrange gave a more precise and general formulation of Maupertius' principle in his book *Analytical Mechanics* published in 1788. He formulated it as

$$\delta S = \delta \int_{t_1}^{t_2} (2T) dt = 0, \tag{9.4.1}$$

where T is the kinetic energy of a dynamical system with the constraint that the total energy, $(T + V)$ is constant along the trajectories, and V is the potential energy of the system. He also derived the celebrated equation of motion for a holonomic dynamical system

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i, \tag{9.4.2}$$

where q_i are generalized coordinates, $\dot{q}_i$ is the velocity, and Q_i is the force. For a conservative dynamical system, $Q_i = -\frac{\partial V}{\partial q_i}$, $V = V(q_i)$, $\frac{\partial V}{\partial q_i} = 0$, then (9.4.2) can be expressed in terms of the Lagrangian, $L = T - V$, as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0. \tag{9.4.3}$$

In 1744, Euler reformulated this principle in a more general way so that it becomes more useful in mathematics and physics. More precisely, the discovery of the calculus of variations in a modern sense began with the independent work of Euler and Lagrange. Indeed, Euler systematically solved a large number of problems of different kinds and discovered a general geometric approach to their solutions.

In the simplest case, Euler's problem is to determine a curve $y = y(x)$ that makes the functional

$$I(y) = \int_a^b F(x, y, y')dx, \qquad y' = \frac{dy}{dx}, \tag{9.4.4}$$

minimum (or maximum), where the function takes prescribed values at the end points fixed, that is, $y(a) = \alpha$ and $y(b) = \beta$; y belongs to the class $C^2([a, b])$ of functions which have continuous derivatives up to second-order in $a \le x \le b$, and F has continuous second-order derivatives with respect to all of its arguments.

Assuming that $I(y)$ has an extremum at some $y \in C^2([a, b])$, then we consider the set of all variations $y + \varepsilon z$ for fixed y, where z is an arbitrary function belonging to $C^2([a, b])$ such that $z(a) = z(b) = 0$. We next consider the increment of the functional

$$\delta I = I(y + \varepsilon z) - I(y) = \int_a^b [F(x, y + \varepsilon z, y' + \varepsilon z') - F(x, y, y')]\, dx. \tag{9.4.5}$$

It follows from (9.4.5) combined with the Taylor expansion of

$$F(x, y + \varepsilon z,\, y' + \varepsilon z') = F(x, y, z) + \varepsilon\left(z\,\frac{\partial F}{\partial y} + z'\,\frac{\partial F}{\partial y'}\right) + \frac{\varepsilon^2}{2!}\left(z\,\frac{\partial F}{\partial y} + z'\,\frac{\partial F}{\partial y'}\right)^2 + \cdots \tag{9.4.6}$$

that

$$I(y + \varepsilon z) = I(y) + \varepsilon\, \delta I + \frac{\varepsilon^2}{2!}\, \delta^2 I + \cdots, \tag{9.4.7}$$

where the first and second variations of I are given by

$$\delta I = \int_a^b \left(z\frac{\partial F}{\partial y} + z'\frac{\partial F}{\partial y'}\right) dx, \tag{9.4.8}$$

$$\delta^2 I = \int_a^b \left(z\frac{\partial F}{\partial y} + z'\frac{\partial F}{\partial y'}\right)^2 dx. \tag{9.4.9}$$

The necessary condition for the functional $I(y)$ to have an extremum (that is, $I(y)$ is stationary at y) is that the first variation becomes zero at y so that

$$0 = \delta I = \int_a^b \left(z\,\frac{\partial F}{\partial y} + z'\,\frac{\partial F}{\partial y'}\right) dx, \tag{9.4.10}$$

which is, by integrating the second integral by parts,

$$\int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] z\,dx + \left[z\,\frac{\partial F}{\partial y'}\right]_a^b = 0.$$

Since $z(a) = 0 = z(b)$, this means that, for any z,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0. \tag{9.4.11}$$

This is the celebrated *Euler* (or the *Euler–Lagrange*) *equation.* In general, this is a nonlinear second-order ordinary differential equation with the boundary conditions that specify the solution curve at the end points $x = a$ and $x = b$. This equation and its many applications included the discovery that the cateroid and the right helicoid are minimal surfaces.

Euler also considered the same problem with the additional constraint

$$\int_a^b G(x, y, y')dx = C, \tag{9.4.12}$$

where C is a constant. This is called *isoperimetric problem* because of the analogy with the classical problem of maximizing the area under a curve when the perimeter is held constant. Thus, the Euler problem (9.4.4) with the constraint condition (9.4.12) is equivalent to the original problem but with F replaced by $F+\lambda G$ where λ is a suitable constant to be determined.

Euler also generalized the original problem when $F = F(x,\, y,\, y^{(1)},\, y^{(2)},$ $\cdots\, y^{(n)})$ with and without constraints. However, in the simple case without constraints except that the end points are fixed, he derived the celebrated equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y^{(1)}}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y^{(2)}}\right) - \cdots + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) = 0. \tag{9.4.13}$$

After determining the function y which makes $I(y)$ stationary, the question of the nature of the extremum arises, that is, its minimum, maximum, or saddle point properties. To answer this question, the second variation defined in (9.4.9) is needed, if terms of $O(\varepsilon^3)$ are neglected in (9.4.9) or if they vanish for case of quadratic F, it follows from (9.4.9) that a necessary condition for the functional $I(y)$ to have a minimum $I(y_0) \le I(y)$ at $y = y_0$ is that $\delta^2 I \ge 0$, for $I(u)$ to have a maximum $I(y_0) \ge I(y)$ at $y = y_0$ is that $\delta^2 I \le 0$ at $y = y_0$ respectively for all admissible values of z. These results enable to determine the upper or lower bounds for the stationary value $I(y_0)$ of the functional.

The Euler–Lagrange variational problem involving two independent variables is to determine a function $u(x, y)$ in a domain $D \subset R^2$ satisfying the boundary conditions prescribed on the boundary ∂D of D and extremizing the functional

$$I[u(x,y)] = \iint_D F(x, y, u, u_x, u_y)\, dx\, dy, \tag{9.4.14}$$

where the function F is defined over the domain D and assumed to have continuous second-order partial derivatives.

Similarly, for functionals depending on a function of two independent variables, the first variation δI of I is defined by

$$\delta I = I(u + \varepsilon v) - I(u). \tag{9.4.15}$$

In view of Taylor's expansion theorem, this reduces to

$$\delta I = \iint_D \left[\varepsilon\left(vF_u + v_x F_p + v_y F_q\right) + 0\left(\varepsilon^2\right)\right] dx\, dy, \tag{9.4.16}$$

where $v = v(x, y)$ is assumed to vanish on ∂D and $p = u_x$ and $q = u_y$.

A necessary condition for the functional I to have an extremum is that the first variation of I vanishes, that is,

$$\begin{aligned} 0 = \delta I &= \iint_D \left(vF_u + v_x F_p + v_y F_q\right) dx\, dy \\ &= \iint_D v\left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right) dx\, dy \\ &\qquad + \iint_D \left[v\left(\frac{\partial}{\partial x}F_p + \frac{\partial}{\partial y}F_q\right) + \left(v_x F_p + v_y F_q\right)\right] dx\, dy \\ &= \iint_D v\left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right) dx\, dy \\ &\qquad + \iint_D \left[\frac{\partial}{\partial x}(vF_p) + \frac{\partial}{\partial y}(vF_q)\right] dx\, dy. \end{aligned} \tag{9.4.17}$$

We assume that boundary curve ∂D has a piecewise, continuously moving tangent so that Green's theorem can be applied to the second double integral in (9.4.17). Consequently, (9.4.17) reduces to

$$\begin{aligned} 0 = \delta I = \iint_D v\left(F_u - \frac{\partial}{\partial x}F_p - \frac{\partial}{\partial y}F_q\right) dx\, dy \\ + \int_{\partial D} v\left(F_p dy - F_q dx\right). \end{aligned} \tag{9.4.18}$$

Since $v = 0$ on ∂D, the second integral in (9.4.18) vanishes. Moreover, since v is an arbitrary function, it follows that the integrand of the first integral in (9.4.18) must vanish. Thus, the function $u(x,y)$ extremizing the functional defined by (9.4.14) satisfies the partial differential equation

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x} F_p - \frac{\partial}{\partial y} F_q = 0. \tag{9.4.19}$$

This is called the *Euler–Lagrange equation* for the variational problem involving two independent variables.

The above variational formulation can readily be generalized for functionals depending on functions of three or more independent variables. Many physical problems require determining a function of several independent variables which will lead to an extremum of such functionals.

Example 9.4.1. Find $u(x,y)$ which extremizes the functional

$$I\left[u(x,y)\right] = \iint_D \left(u_x^2 + u_y^2\right) dx\, dy, \quad D \subset R^2. \tag{9.4.20}$$

The Euler–Lagrange equation with $F = u_x^2 + u_y^2 = p^2 + q^2$ is

$$\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial p}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial q}\right) = 0,$$

or

$$u_{xx} + u_{yy} = 0. \tag{9.4.21}$$

This is a two-dimensional Laplace equation. Similarly, the functional

$$I\left[u(x,y,z)\right] = \iiint_D \left(u_x^2 + u_y^2 + u_z^2\right) dx\, dy\, dz, \quad D \subset R^3 \tag{9.4.22}$$

will lead to the three-dimensional Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0. \tag{9.4.23}$$

In this way, we can derive the n-dimensional Laplace equation

$$\nabla^2 u = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n} = 0. \tag{9.4.24}$$

Example 9.4.2. *(Plateau's Problem).* Find the surface S in (x,y,z) space of minimum area passing through a given plane curve C.

The direction cosine of the angle between the z-axis and the normal to the surface $z = u(x,y)$ is $\left(1 + u_x^2 + u_y^2\right)^{-\frac{1}{2}}$. The projection of the element dS of the area of the surface onto the (x,y)-plane is given by $\left(1 + u_x^2 + u_y^2\right)^{-\frac{1}{2}} dS = dx\, dy$. The area A of the surface S is given by

$$A = \iint_D \left(1 + u_x^2 + u_y^2\right)^{\frac{1}{2}} dx\, dy, \tag{9.4.25}$$

where D is the area of the (x,y)-plane bounded by the curve C.

The Euler–Lagrange equation with $F = \left(1 + p^2 + q^2\right)^{\frac{1}{2}}$ is given by

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1+p^2+q^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1+p^2+q^2}}\right) = 0. \qquad (9.4.26)$$

This is the *equation of minimal surface*, which reduces to the nonlinear elliptic partial differential equation

$$\left(1 + u_y^2\right) u_{xx} - 2u_x u_y u_{xy} + \left(1 + u_x^2\right) u_{yy} = 0. \qquad (9.4.27)$$

Therefore, the desired function $u(x, y)$ should be determined as the solution of the *nonlinear Dirichlet problem* for (9.4.27). In general, this is difficult to solve. However, if the equation (9.4.26) is linearized around the zero solution, the square root term is replaced by one, and then the Laplace equation is obtained.

Although Euler and Lagrange laid the solid mathematical foundation of the calculus of variations geometrically as well as analytically, the subject had received tremendous attention of Carl G. Jacobi, and Sir William Hamilton, and then of Karl Weierstrass who first gave a more modern foundation of the theory in his lectures at Berlin between 1865 and 1890 and thus inaugurated the modern period of the calculus of variations. This period ended gloriously when the calculus of variations burst into full flower with the advent of functional analysis, and the 1910 work of Jacques Hadamard, David Hilbert and Hermann Weyl (1885-1955). During the earlier period, the problems of calculus of variations were reduced to questions of the existence of solutions of ordinary and partial differential equations until Hilbert developed a new method in which the existence of a minimizing function was established as the limit of a sequence of approximations. At the conclusion of his famous lecture on ‘Mathematical Problems’ at the Paris International Congress of Mathematicians in 1900, he assumed the path to be extremal of $I(y)$ and Hilbert then formulated another derivation of Weierstrass’ excess function and a new approach to Jacobi’s problem of determining necessary and sufficient conditions for the existence of a minimum of functional $I(y)$ and all this without the introduction of the second variation of I. When a function is over a convex set of constraints, the calculus of variations led to the theory of variational inequality. In this case, the classical Euler equations have been replaced by a set of inequalities. Subsequently, Hartman and Stampacchia (1966) established a very basic variational inequality.

Finally, the calculus of variations entered the new and wider field of *global* problems with the modern work of G. D. Birkhoff and Marston Morse

(1892-1977). They succeeded in liberating the theory of calculus of variations from the limitations imposed by the restriction to 'small variations', and gave a general treatment of the global theory of the subject with 'large variations'. This is followed by the Morse-Smale Index theorem for a global analysis of variational problems in several variables with applications to the computation of the Morse-Smale Index of the catenoid and Enneper's minimal surface. Subsequently, the calculus of variations led to a new modern area known as the *Morse Theory*, with ramifications in geometry, analysis and topology, which has become a major part of mathematics and mathematical physics of the twentieth century.

LEONARD EULER

Des Academies Royales des Sciences de Paris, de Londres, de Berlin, de Petersb[illegible] &c Né à Basle, le 15 Avril 1707, Mort à St. Petersbourg, le 18 de Septembre 1783.

Dessiné par Madᵉ. du Piery, d'après le Medaillon envoyé à l'Acad. des Sciences par l'Académie de Petersbourg. *Dupin sculp.*

A Paris, chez Esnauts *et* Rapilly, *rue St. Jacques, à la Ville de Coutances.* Avec Priv. du Roi

Chapter 10

The Euler Equations of Motion in Fluid Mechanics

> "True Laws of Nature cannot be linear."
>
> *Albert Einstein*

> "... the progress of physics will to a large extent depend on the progress of nonlinear mathematics, of method to solve nonlinear equations ... and therefore we can learn by comparing different nonlinear problems."
>
> *Werner Heisenberg*

10.1 Introduction

Euler's major work in the field of fluid mechanics was essentially based on the continuum hypothesis and Newton's laws of motion. However, his work provided the basic foundation of mathematical theory of fluid mechanics which was surrounded by his discovery of the calculus of variations as well as partial differential equations. He made fundamental contributions to hydrostatics and hydrodynamics during the period of 1752-1761 and published several major articles in these fields in the *Mémories de l'Academie des Sciences de Berlin* in 1757. The first of these papers dealt with the basic general concepts, principles and equilibrium equations of fluid. The second and the third papers were basically concerned with the *equation of conservation of mass* (or the *continuity equation*) and the Euler nonlinear equations of motion of compressible fluid flows. Subsequently, he formulated the equations of motion and the continuity equation for an inviscid, incompressible fluid flows with the first proof of the famous d'Alembert paradox in an inviscid fluid flow past a rigid body. Historically, considerable progress was made on theoretical fluid mechanics during the 18th century by Jean d'Alembert, Daniel Bernoulli, Alexis C. Clairaut, and Joseph Lagrange. Among these great mathematical scientists, Euler made the most fundamental contributions to fluid mechanics by establishing his famous equations of motion. To celebrate Euler's great contributions to mechanics, G. K. Mikhailov (2007) writes: "Euler possessed a rare gift of systematizing and generalizing scientific ideas, which allowed him to present large parts of mechanics in a relatively definite form."

Based on his evaluation of Euler's published papers and books, and unpublished notebook written in 1725-1727, G. K. Mikhailov (1957) also gave a new and surprising insight into Euler's remarkable work on the theoretical development of fluid mechanics and hydraulics and states:

"It is generally known that the creation of the foundations of modern hydrodynamics of ideal fluids is one of the fruits of Euler's scientific activity. Less well known is his role in the development of theoretical hydraulics, that is, as usually understood, the hydrodynamic theory of fluid motion under a one-dimensional flow model. Traditionally — and with good reason — it is assumed that the foundations of hydraulics were developed by Daniel and Johann Bernoulli in their works published between 1729 and 1743. In fact, during the second quarter of the eighteenth century Euler did not publish even a single paper on the elements of hydraulics. The central theme of most of the recent historical-critical studies on the state of hydraulics in that period is the determination of the respective contributions of Daniel and Johann Bernoulli. But Euler stood, all this time, just beyond the curtain of the stage on which the action was taking place, although almost no contemporary was aware of that."

10.2 Eulerian Descriptions of Fluid Flows

It was Euler who gave the first formulation of Eulerian and Lagrangian descriptions of fluid flows. In the Eulerian description, the fluid flow is specified by the velocity field $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ as a function of position $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and time t. In the Lagrangian framework, the fluid motion is described by the position $\mathbf{x} = \mathbf{x}(\mathbf{s}, \tau)$ of a fluid particle as a function of and time τ, where the initial position $\mathbf{r} = \mathbf{x}(\mathbf{s}, \tau = 0)$ is used as a label. The label coordinates $\mathbf{s} = (a, b, c)$ are assumed to form a continuous three-dimensional manifold. Thus, the Lagrangian description of the fluid flow represents a time-dependent mapping from label (initial position) space to position space. The fluid velocity is given by

$$\mathbf{u}(\mathbf{s},\tau) = \frac{\partial}{\partial \tau}\,\mathbf{x}(\mathbf{s},\tau), \tag{10.2.1}$$

and the acceleration of the fluid is given by

$$\mathbf{a}(\mathbf{s},\tau) = \frac{\partial}{\partial \tau}\,\mathbf{u}(\mathbf{s},\tau) = \frac{\partial^2}{\partial \tau^2}\,\mathbf{x}(\mathbf{s},\tau). \tag{10.2.2}$$

The Eulerian and the Lagrangian description differ in their choice of dependent and independent variables. However, the two descriptions are equivalent. To transform from the Lagrangian description $\mathbf{x} = \mathbf{x}(\mathbf{s},\tau)$ to the Eulerian description $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$, we have to perform the following operations:

(i) differentiate $\mathbf{x} = \mathbf{x}(\mathbf{r},\tau)$ with respect to τ to find $\mathbf{u} = \mathbf{u}(\mathbf{r},\tau)$,
(ii) invert $\mathbf{x} = \mathbf{x}(\mathbf{r},\tau)$ to derive $\mathbf{r} = \mathbf{r}(\mathbf{x},\tau)$, and
(iii) put $\mathbf{r} = \mathbf{r}(\mathbf{x},t)$ into $\mathbf{u} = (\mathbf{r},\tau)$ to obtain $\mathbf{u} = (\mathbf{x},t)$.

On the other hand, to transform from the Eulerian to the Lagrangian description, it is necessary to solve the ordinary differential equations

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}(t),t) \tag{10.2.3}$$

with the initial data $\mathbf{x}(t=0) = \mathbf{r}$ for all $\mathbf{r}$.

Thus, in modern language, the fundamental quality of the Eulerian description is the velocity gradient tensor, whereas the main quantity of the Lagrangian description is the deformation tensor.

Euler first introduced the concept of the *material* (or *convective* or *total derivative*) in 1770 in the form

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + u_i \phi_{,i} = \frac{\partial \phi}{\partial t} + (\mathbf{u}\cdot\nabla)\,\phi \tag{10.2.4}$$

where $\phi_{,i} = \partial\phi/\partial x_i$.

When $\mathbf{u}$ is known as a function of $\mathbf{x}$ and t, expression (10.2.4) enables us to compute $(D\phi/Dt)$ as a function of $\mathbf{x}$ and t. As such, formula (10.2.4) represents the *material derivative in the spatial form.* Note that the first term on the right hand side of (10.2.4), namely, $(\partial\phi/\partial t)$ represents the *local rate of change of* ϕ and the second term, $u_i\,\phi_{,i} = (\mathbf{u}\cdot\nabla)\,\phi$ is the contribution due to the motion. This second term is usually referred to as the *convective rate of change of* ϕ.

It can be verified that the *material derivative operator*

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_i(\ )_{,i} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \tag{10.2.5}$$

which operators on functions represented in spatial form, satisfies all the rules of partial differentiation.

We next define the *acceleration in spatial form* by substituting $\phi = u_i$ in (10.2.4) so that

$$\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_k u_{i,k}. \tag{10.2.6a}$$

Or,

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u}. \tag{10.2.6b}$$

When $\mathbf{u}(\mathbf{x}, t)$ is known as a function of $\mathbf{x}$ and t, expression (10.2.6b) determines $(D\mathbf{u}/Dt)$ directly in terms of $\mathbf{x}$ and t; this expression (10.2.6b) serves as a *formula for acceleration in the spatial form.*

Using the standard vector identity

$$(\mathbf{u} \cdot \nabla)\,\mathbf{u} = \frac{1}{2}\nabla(u^2) - \mathbf{u} \times (\mathrm{curl}\mathbf{u}) = \frac{1}{2}\nabla(u^2) + (\mathrm{curl}\mathbf{u}) \times \mathbf{u}, \tag{10.2.7}$$

where $\mathbf{u} \cdot \mathbf{u} = u^2$, the formula (10.2.6b) can be written in the form

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla(u^2) + \boldsymbol{\omega} \times \mathbf{u}, \tag{10.2.8}$$

where $\boldsymbol{\omega} = \mathrm{curl}\mathbf{u}$ is the *vorticity vector*.

It follows from (10.2.6b) and (10.2.8) that the acceleration vector is made up of two parts:

(i) the *local rate of change of velocity*, namely, $\left(\frac{\partial \mathbf{u}}{\partial t}\right)$, and
(ii) the *convective rate of change of velocity*, $(\mathbf{u} \cdot \nabla)\,\mathbf{u} = \frac{1}{2}\nabla(u^2) + \boldsymbol{\omega} \times \mathbf{u}$.

Evidently, the second part is *quadratically nonlinear* in nature. Thus, the acceleration depends quadratically on the velocity field, and a given motion *cannot* be represented as a superposition of two independent motions in general.

Euler first (1757) also formulated the equation of motions for a compressible fluid with variable density, $\rho\,(\mathbf{x}, t)$ in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\frac{1}{\rho}\,\nabla p, \tag{10.2.9}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{10.2.10a}$$

Or, equivalently, this mass conservation can be expressed as

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \mathbf{u} = 0. \tag{10.2.10b}$$

These equations (10.2.9)–(10.2.10) constitute four scalar nonlinear partial differential equations in five unknowns u, v, w, p and ρ so that they need to be supplemented with a fifth equation expressing the compressibility properties of the fluid. They are of widespread utility because they describe the phenomena of propagation of sound waves, as well as being needed to study any flows of air at speeds of order of $100/ms$. Indeed, the phenomena of sound (or acoustic waves) depend on the compressibility property of a fluid.

Based on Newton's Second Law of motion to an inviscid fluid, Euler (1761) combined the rate of change of total momentum with the internal pressure and then first formulated his celebrated equations of motion and the continuity (mass conservation) equation for an inviscid, incompressible fluid in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho}\, \nabla p + \mathbf{F}, \tag{10.2.11}$$

$$div\, \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{10.2.12}$$

where $\mathbf{u}\,(x,t) = (u,v,w)$ is the velocity vector and $p\,(\mathbf{x},t)$ is the pressure field at the point $\mathbf{x} = (x,y,z)$ and time t, ρ is the constant density and $\mathbf{F}$ is the external force field. These equations (10.2.11) and (10.2.12) constitute a closed system of four nonlinear partial differential equations with four unknowns u, v, w and p. So these equations with appropriate initial and boundary conditions are sufficient to determine the velocity field $\mathbf{u}$ and the pressure p uniquely. The Euler equations of incompressible fluid flow are widely used for an understanding of observed phenomena of vortex motion which occurs in a regime where fluid is considered to be incompressible. He also applied (10.2.11)–(10.2.12) to blood flow in arteries in the human body. d'Alembert also derived these equations (10.2.11)–(10.2.12) in 1752. In his revolutionary ballistic research work in response to a royal assignment from Frederick the Great in 1745, Euler derived the first mathematical proof of the famous d'Alembert's paradox in fluid mechanics that an inviscid potential flow around a rigid body moving with at a uniform speed exerts no resistance force on the body, and also provided a pioneering mathematical analysis of subsonic and supersonic air resistances.

More importantly, the Euler equations (10.2.11)–(10.2.12) are widely used to develop the theory of water waves in oceans under the action of the body force $\mathbf{F} = -g\rho\widehat{\mathbf{k}}$, g is the acceleration of gravity, $\widehat{\mathbf{k}}$ is the unit vector in the positive z direction. Consequently, the fundamental equations of motion of water waves are (see Debnath (1994)) given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\frac{1}{\rho}\, \nabla p - g\widehat{\mathbf{k}}, \tag{10.2.13}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{10.2.14}$$

In problems of water waves, the motion may be taken as irrotational, which physically means that the individual fluid particles do not rotate. Mathematically, this implies that vorticity $\boldsymbol{\omega} = \mathit{curl}\, \mathbf{u} = \mathbf{0}$ so that there exists a single-valued velocity potential ϕ such that $\mathbf{u} = \nabla\phi$. The continuity equation (10.2.14) then reduces to the Laplace equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{10.2.15}$$

So, the velocity potential ϕ is a harmonic function. This is, indeed, a great advantage because the velocity field u can be derived from a single potential function ϕ which satisfies the linear Laplace equation (10.2.15). So, the velocity potential ϕ is a harmonic function. This is, indeed, a great advantage because the velocity field u can be derived from a single potential function ϕ which satisfies the linear Laplace equation (10.2.15).

Representing the free surface S of water by $S\,(x, y, z, t) = \eta\,(x, y, t) - z = 0$, the normal velocity of the surface is $-S_t/\,|\nabla S|$ which is equal to the normal velocity of the fluid, $\mathbf{u} \cdot \mathbf{n} = \mathbf{u} \cdot (\nabla S/\,|\nabla S|)$ so that

$$\frac{DS}{Dt} \equiv \frac{\partial S}{\partial t} + (u \cdot \nabla)\, S = 0 \tag{10.2.16}$$

which is, in terms of the free surface elevation η and the potential $\phi\,(\mathbf{u} \equiv \nabla\phi)$,

$$\eta_t + \phi_x\eta_x + \phi_y\eta_y - \phi_z = 0 \quad \text{on} \quad z = \eta\,(x, y, t)\,. \tag{10.2.17}$$

This is called the *kinematic free surface condition.*

Using a formula $(\mathbf{u} \cdot \nabla)\, \mathbf{u} = \frac{1}{2}\nabla u^2 - \mathbf{u} \times \boldsymbol{\omega}$ combined with $\mathbf{u} = \nabla\phi$, the Euler equation (10.2.13) can be rewritten in the form

$$\nabla \left[\phi_t + \frac{1}{2}\,(\nabla\phi)^2 + \frac{p}{\rho} + gz\right] = 0. \tag{10.2.18}$$

This can be integrated with respect to the space variables to obtain the equation

$$\phi_t + \frac{1}{2}\,(\nabla\phi)^2 + \frac{p}{\rho} + gz = C\,(t)\,, \tag{10.2.19}$$

where $C\,(t)$ is an arbitrary function of time only $(\nabla C \equiv 0)$ determined by the pressures imposed at the boundaries of the fluid flow. Since only the pressure gradient affects the flow, a function of t above added to the pressure field p has no effect on the motion. So, without loss of generality, we can set $C(t) \equiv 0$ in (10.2.19) and consequently, equation (10.2.19) becomes

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} + gz = 0. \tag{10.2.20}$$

This is called *Bernoulli's equation* which determines the pressure in terms of the velocity potential ϕ. In particular, equation (10.2.20) yields the following equation for pressure p at every point of the free surface $z = \eta(x, y, t)$:

$$\phi_t + g\eta + \frac{1}{2}(\nabla\phi)^2 + \frac{p}{\rho} = 0 \quad \text{on} \quad z = \eta(x, y, t). \tag{10.2.21}$$

On the other hand, the normal velocity of fluid must vanish on a solid boundary surface, that is,

$$\mathbf{n} \cdot \mathbf{u} = \mathbf{n} \cdot \nabla\phi \equiv \frac{\partial\phi}{\partial n} = 0. \tag{10.2.22}$$

In particular, at a rigid bottom surface $z = -h(x, y)$, the bottom boundary condition is given by

$$\phi_z + \phi_x h_x + \phi_y h_y = 0 \quad \text{at} \quad z = -h(x, y). \tag{10.2.23}$$

In case of the problem of water waves in an ocean of constant depth h with a rigid bottom, the bottom boundary condition is

$$\phi_z = 0 \quad \text{at} \quad z = -h. \tag{10.2.24}$$

It should be noted that a *single linear boundary condition* (10.2.22) is required on a fixed boundary surface. However, two coupled nonlinear surface conditions are to be prescribed on the free surface $(z = \eta)$ because the free surface elevation function $\eta(x, y, t)$ is involved as an additional unknown function.

With prescribed atmospheric pressure and given initial conditions, equations (10.2.15), (10.2.17), (10.2.21), (10.2.23) or (10.2.24) are sufficient to determine the wave motion in water. These represent the well-known nonlinear system of equations of classical water waves. They can be mathematically derived from the modern variational principle (see Debnath (1994)) for three-dimensional water waves

$$\delta I = \delta \iint_D L\, d\mathbf{x}\, dt = 0, \tag{10.2.25}$$

where the Lagrangian L is assumed to be equal to the pressure so that

$$L = -\rho \int_{-h(x,y)}^{\eta(\mathbf{x},t)} \left[\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz\right] dz, \tag{10.2.26}$$

where D is an arbitrary region in the $(\mathbf{x}, t)$ space, and $\phi(x, y, z, t)$ is the velocity potential of an unbounded fluid lying between the rigid bottom $z = -h(x, y)$ and the free surface $z = \eta(x, y, t)$.

There is no doubt that the Euler equations in both incompressible and compressible fluids have provided the fundamental basis for both classical and modern fluid mechanics. Several major nonlinear evolution equations including the Boussinesq equations, and the Korteweg–de Vries (KdV) equation have been derived from the Euler equations (see Debnath (2005)). During the second half of the twentieth century, many modern evolution equations including the Kadomtsev–Petviashvili (KP) equation, the nonlinear Schrödinger (NLS) equation, the Johnson equation, and Davey–Stewartson equations (see Debnath (2005)) have also been derived from the Euler equations.

Although the Euler equations are over 250 years old, the existence as well as uniqueness of solutions is still an unsolved challenging problem. Another related open question is the behavior of solutions at a finite time. Many numerical simulations of the three dimensional Euler equations have been carried out. Solutions are found to behave very wildly. So, it is very difficult to determine whether a numerical study indicates a breakdown (or instability) of solutions at a finite time. In his original paper, J. T. Stuart (1987) provided a new insight into the possibility of solutions of the nonlinear Euler equations developing a singularity in a finite time. His study of the problem of the evolution of wave motion in a thin boundary layer or shear flow, particularly in relation to the development of turbulence (see also Moffatt, 1985), is closely associated with the natural presence of three-dimensional vortex structures in the flow. These longitudinal vortices in the flow, whose axes are aligned parallel to the main stream, suffer from convection, stretching, and tilting and produce thereby local shear layers which becomes more and more intense as $t \to \infty$. These nonlinear features of the flow lead to the possibility that the flow field can develop a singularity in a finite time. In other words, the solutions of the Euler equations in an inviscid flow under appropriate initial conditions may become singular in a finite time. The findings of Stuart have already received a strong support from somewhat related and independent works of Calogero (1984), Russell and Landahl (1984). Because singularities (see Stuart (1987)) cannot be ruled out, the mathematical question of the blow up problem in the Euler equations is whether singularities can arise in finite time from smooth initial velocities with finite kinetic energy. This is a major unsolved problem of the nonlinear partial differential equations theory. Physically, it makes sense to admit solutions with singularities in them, the concept of solution can possibly be extended to that of weak solution of the Euler equations in three dimensions. Shnirelman (1977) has demonstrated the nonuniqueness of the weak solution of the three dimensional Euler equations. However, this problem has not yet completely been solved. Thus, the study of the Euler equations, which constitute the fundamental equations of modern fluid mechanics, is still one of the most challenging and meaningful problems in the nonlinear partial differential equations.

Subsequently, considering internal processes that lead to energy dissipation, in 1822, Louis M. H. Navier (1785-1836), and in 1845, George G. Stokes (1819-1903) formulated a celebrated equation, universally known as the *Navier–Stokes equations*

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{F} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} \tag{10.2.27}$$

together with the continuity equation (10.2.12), where $\nu > 0$ represents the kinematic viscosity which is constant at constant temperature. Obviously, their starting point was Euler's equations of motion of an inviscid incompressible fluid which led them to generalize Euler's equations with viscosity. However, there are certain major difficulties associated with the three-dimensional Navier–Stokes equations as there are no general results for these equations on the existence of solutions, uniqueness, regularity, and continuous dependence of solutions on the initial data. This remains as the challenging unsolved problems of the 21st century.

In terms of some representative length scale ℓ, and velocity scale U, it is convenient to introduce the nondimensional flow variables

$$\mathbf{x}^* = \frac{\mathbf{x}}{\ell}, \quad \mathbf{t}^* = \frac{Ut}{\ell}, \quad \mathbf{u}^* = \frac{\mathbf{u}}{U}, \quad \mathbf{p}^* = \frac{p}{\rho U^2}. \tag{10.2.28}$$

In terms of these nondimensional variables, equation (10.2.27) without the external force ($\mathbf{F} = \mathbf{0}$), can be written, dropping the astericks, as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} = -\nabla p + \frac{1}{R} \nabla^2 \mathbf{u}, \tag{10.2.29}$$

where $R = (U\ell/\nu)$ is universally called the *Reynolds number*. This is one of the most fundamental dimensionless parameters needed for the specification of the dynamical state of viscous flow fields with geometrically similar boundary and initial conditions. Physically, it measures the ratio of inertial forces of order $\left(U^2/\ell\right)$ to viscous forces of order $\left(\nu U/\ell^2\right)$, and hence it has a special physical significance.

There are certain major difficulties associated with the Navier–Stokes equations. First, there are no general results for the Navier–Stokes equations on existence, uniqueness, regularity, and continuous dependence on the initial conditions. Second, there are indications that solutions of the three-dimensional Navier–Stokes equations can be singular at certain places and at certain times in the flow. Third, in view of the strong nonlinear convective term in (10.2.27) or (10.2.29), there is no general analytical method of solution for an arbitrary Reynolds number, R (or viscosity ν).

In absence of the external forces ($\mathbf{F} = \mathbf{0}$), it is preferable to write equation (10.2.27) in the form (since $\mathbf{u} \times \boldsymbol{\omega} = \frac{1}{2}\nabla u^2 - \mathbf{u} \cdot \nabla \mathbf{u}$, $u^2 = \mathbf{u} \cdot \mathbf{u}$)

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{p}{\rho} + \frac{1}{2} u^2 \right) - \mathbf{u} \cdot \nabla \mathbf{u}. \tag{10.2.30}$$

We can eliminate the pressure from equation (10.2.30) by taking the curl of the equation (10.2.30) so that it becomes

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = curl\,(\mathbf{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}, \tag{10.2.31}$$

which reduces to, by $div\,\mathbf{u} = 0$ and $div\,curl \mathbf{u} = 0$,

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u} - (\mathbf{u} \cdot \nabla)\,\boldsymbol{\omega} + \nu \nabla^2 \boldsymbol{\omega}. \tag{10.2.32}$$

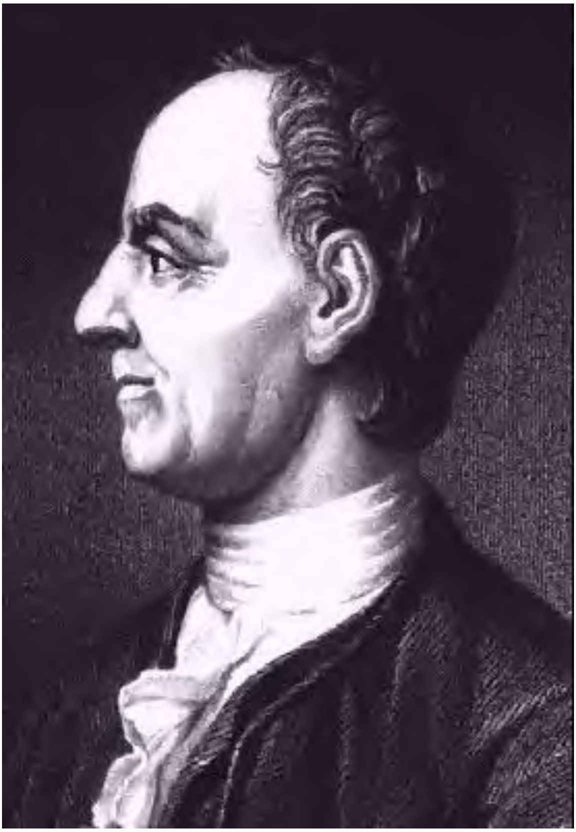
Or equivalently,

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \tag{10.2.33}$$

This is known as the *vorticity transport equation*. The left hand side of (10.2.32) represents the rate of change of vorticity. The first two terms on the right-hand side represent the rate of change of vorticity due to stretching and twisting of vortex lines. The last viscous term describes the diffusion of vorticity by molecular viscosity.

Mathematically, the Navier–Stokes equations (10.2.27) or (10.2.29) are valid for all values of ν (or R). In the limit as $\nu \to 0$ (or $R \to \infty$), the second-order Navier–Stokes equations reduce to the first-order Euler equations (10.2.11) with $\mathbf{F} = \mathbf{0}$. So, the Navier–Stokes equations lead to a singular perturbation problem. In the limit of zero viscosity ($R \to \infty$), there exist thin layers near the boundary, where the significant change from a viscosity dominated behavior to an inviscid behavior takes place, or at least that is the physical nature in quiescent situations near flat or curved boundaries. In other words, the effect of viscosity is dominant within the boundary layers, however small the viscosity ν may be, and hence, the linear viscous term in the Navier–Stokes equations must be retained to satisfy all necessary boundary conditions. In classical inviscid theory ($\nu \to 0$ or $R \to \infty$), the boundary layer does not exist, so that an inviscid fluid is free to slip on the surface of the solid body, and only the normal component of the velocity is zero on the surface. However, no general and rigorous mathematical proof of Ludwig Prandtl's (1875-1953) 1910 remarkable discovery of boundary layer hypothesis has been available yet, but it is strongly supported by many experimental observations of particular fluid flow models, and hence, the Prandtl hypothesis has been applied successfully to many different kinds of flow field. So, boundary layers are very real but difficult to prove rigorously, especially, for curved domains and for time-dependent fluid flows. At present, the problems of the zero viscosity limit is still unsolved in two space dimensions. Recently, Barenblatt and Chorin (1997, 1998) raised questions about the physical problems of boundary layers and challenged the classical Prandtl boundary layer theory. Indeed, in general, the double limits $t \to \infty$ $\nu \to 0$ involved in solutions of the Navier–Stokes do not commute. This is most clearly observed for the case of two dimensional Navier–Stokes equations without external force field. In this case, any smooth solution of the Euler equations is a finite-time inviscid limit, and the infinite-time inviscid limit is *unique*. The study of the zero-viscosity long-time statistics including the Kolmogorov spectrum is still an open problem. In his 1934 pioneering work, Jean Leray (1906-1998) proved the existence of global weak solutions of the three dimensional Navier–Stokes equations. It was shown by Caffarelli et al. (1982) that singularities, if any, of the Navier–Stokes are confined to a space-time set of dimension less than one, but their uniqueness has not yet been proved. The existence of weak solutions of the Navier–Stokes equations that dissipate energy by constant flux of energy, in the whole space, in the correct function space has not yet been established. The study of the zero viscosity limit ($\nu \to 0$ or $R \to \infty$) in bounded domains with boundaries is also incomplete.

Finally, it is interesting to point out that the Navier–Stokes equations agree well with experiments in real fluids under many and varied situations. Numerical simulations of the three-dimensional Navier–Stokes equations exhibit no evidence of breakdown (or instability). The Euler equations are simply the limiting cases of zero viscosity of the Navier–Stokes equations. However, solutions of the Euler equations behave very differently from solutions of the Navier–Stokes equations, even when viscosity is very small.



Chapter 11

Euler's Contributions to Mechanics and Elasticity

> "It is precisely on this principle that all other principles must be based, such as those already used to determine the motion of rigid and fluid bodies in mechanics and hydraulics and those as yet unknown that will be required in order to deal with both the aforementioned cases of rigid bodies, and the many other cases concerning fluid bodies."

Leonhard Euler

> "Euler's general and final statement of the principles of linear momentum and moment of momentum", representing "*the fundamental, general, and independent laws of mechanics*, for all kinds of motions of all kinds of bodies."

Clifford Truesdell

11.1 Introduction

Euler made some landmark contributions to the mechanics of particles and of flexible elastic bodies and elasticity. Based on Newton's remarkable work on mechanics, Euler embarked on a major study of particle dynamics and dynamics of rigid bodies which were published in his first two large volumes on *Mechanica* in 1736. In his memoir "Discovery of a new principle of mechanics", presented to the Berlin Academy in 1750, Euler described the principle he had discovered — "a general and fundamental principle of the whole of mechanics." In this memoir, Euler went further to say that: "It is precisely on this principle that all other principles must be based, such as those already used to determine the motion of rigid and fluid bodies in mechanics and hydraulics and those as yet unknown that will be required in order to deal with both the aforementioned cases of rigid bodies, and the many other cases concerning fluid bodies."

In the Preface of his first volume, Euler expressed his views of Newton's synthetic method as follows:

"However if analysis is needed anywhere, then it is certainly in mechanics. Although the reader can convince himself of the truth of the exhibited propositions, he does not acquire a sufficiently clear and accurate understanding of them, so that if those questions be ever so slightly changed, he will not be able to answer them independently, unless he turn to analysis and solve the same propositions using analytic methods. This in fact happened to me when I began to familiarize myself with Newton's *Principia* and Hermann's *Phoronomia*; although it seemed to me that I clearly understood the solutions of many of the problems, I was nevertheless unable to solve problems differing slightly from them. But then I tried, as far as I was able, to distinguish the analysis [hidden] in the synthetic method and to my own ends rework analytically those same propositions, as a result of which I understood the essence of each problem much better. Then in the same manner I investigated other works relating to this science, scattered here, there and everywhere, and for my own sake I expounded them [anew] using a systematic and unified method, and re-ordered them more conveniently. In the course of these endeavors not only did I encounter a whole series of problems hitherto never even contemplated — which I have very satisfactorily solved —, but I discovered many new methods thanks to which not only mechanics, but analysis itself, it would seem, has been significantly enriched. It was thus that this essay on motion arose, in which I have expounded using the analytic method and in convenient order both that which I have found in others' works on the motion of bodies and what I myself have discovered as a result of my ruminations."

Unlike the work of his predecessors, his method in this area were distinguished by the rigorous and systematic application of mathematical analysis. He provided the basic foundation of analytic mechanics. He first gave a remarkable mathematical analysis of the kinematics and dynamics of a point mass both in vacuum and in resisting medium. His study of motion of a point mass under a central force represents an extension of Newton's *Principia* and became a remarkable introduction to his further subsequent studies in celestial mechanics and astronomy. In 1764, Euler published his great third volume of *Mechanica* entitled *Theory of Motion of Rigid Bodies*. In these three volumes of work, he first formulated the equations of motion of a particle and then applied them systematically in finding analytical solutions of many dynamical problems. More importantly, he first made stability analysis of solid elastic bodies in various configurations with critical parameters determining loss of stability. In order to describe Euler's famous *Mechanica*, J. L. Lagrange's praise of Euler is worth quoting: "it must be acknowledged as the first substantial work in which analysis was applied to study of motion."

In spite of Newton's remarkable work on mechanics, in 1752, Euler formulated the complete general principle of linear momentum which states that the total force on a solid body is equal to the rate of change of the total momentum of the body. Later in 1775, he also discovered the principle of moment of momentum (that is, the angular momentum) which asserts that the total torque on a body about some fixed point is equal to the rate of change of the moment of momentum of the body about the same point. In his essay entitled *A new method of determining the motion of rigid bodies and Euler's formula* presented at the St. Petersburg Academy in 1775, Euler announced for the first time the six equations of motion of a rigid body representing the laws governing the linear momentum and the angular momentum. In 1968, Clifford Truesdell (1919-2000) described the laws represented by these equations "Euler's general and final statement of the principles of linear momentum and moment of momentum", representing "*the fundamental, general, and independent laws of mechanics*, for all kinds of motions of all kinds of bodies".

Historically, the classical theory of elasticity is concerned with the study of deflection of elastic beams in various geometrical configurations and this illustrates many new kinds of phenomenon. The bending of a beam of arbitrary cross section by end couples is regarded as one of the oldest branches of the theory of elastic stability which is of special mathematical interest. Euler's famous investigation of the stability of an initially straight elastic beam under progressive end loads provided a complete analysis of the behavior of the beam after buckling has occurred. Euler's theory remained for nearly two centuries the only complete investigation of the post buckling behavior. The fundamental law describes the *curvature* κ (or the *radius of curvature*, R where $\kappa = \frac{1}{R}$) of a beam bent by couples of magnitude M (often called the *bending moment*) applied to its ends so that M is constant along the beam. This law, known as the *Euler–Bernoulli law*, is expressed mathematically by

$$M = \frac{EI}{R} = \kappa\, EI, \tag{11.1.1}$$

where E is called *Young's modulus of elasticity* and I is called the *moment* *of inertia* of the cross section of the beam. The product EI is called the *flexural rigidity* of the beam. In an actual beam, M is a function of x and $y(x)$ is the deflection of the beam. Equation (11.1.1) represents the curvature at a point of the beam in terms of the bending moment at that point. The radius of curvature R is given in terms of the deflection $y(x)$ by

$$\kappa = \frac{1}{R} = \frac{y''}{(1+y'^2)^{3/2}}. \tag{11.1.2}$$

In problems where the slope, y' of the beam is small and y'^2 is negligible, so that $\kappa(x) \sim y''$ and (11.1.1) can be replaced by

$$M(x) = EIy''. \tag{11.1.3}$$

This equation can easily be solved under appropriate boundary conditions. However, for problems of large deflections, the equation (11.1.1) is a complete nonlinear equation which is difficult to solve except for some simple cases.

This chapter is devoted to Euler's major contributions to solid mechanics and elasticity. Special attention is given to some remarkable impact of the Euler angle formulation, the Euler axis formulation and the Euler–Rodrigues quaternion formulation on modern aerodynamics, in general and aircraft dynamics, in particular.

11.2 Euler's Work on Solid Mechanics

Euler first introduced the term, *moment of inertia* as well as the existence of principal axes and moment of inertia of a rigid body. This was followed by his investigation of motion of a rigid body about a fixed point in it under no force. With A, B, C as the principal moments of inertia of a rigid body about the three perpendicular axes fixed in it and p, q, r as the resolved parts of the angular velocities about the principal axes, Euler gave the equations of motion of the body

$$A\,\frac{dp}{dt} = (B-C)qr, \tag{11.2.1}$$

$$B\,\frac{dq}{dt} = (C-A)pr, \tag{11.2.2}$$

$$C\,\frac{dr}{dt} = (A-B)pq. \tag{11.2.3}$$

The equation for conservation of energy is given by

$$\frac{1}{2}\left(Ap^2 + Bq^2 + Cr^2\right) = E, \tag{11.2.4}$$

where E denotes the total energy. He also wrote the equation

$$A^2p^2 + B^2q^2 + C^2r^2 = M^2, \tag{11.2.5}$$

where M^2 denotes the total angular momentum about the fixed point and M is a constant, as there is no external force and the reaction passes through the fixed point.

Expressing q^2, r^2 in terms of p^2, we obtain

$$q^2 = \alpha_1 - \alpha_2\, p^2, \qquad r^2 = \beta_1 - \beta_2\, p^2,$$

where

$$\alpha_1 = \frac{M^2 - 2EC}{B(B-C)}, \quad \alpha_2 = \frac{A(A-C)}{C(B-C)}, \quad \beta_1 = \frac{M^2 - 2EB}{C(C-B)}, \quad \beta_2 = \frac{A(A-B)}{C(C-B)},$$

Substituting in (11.2.1), we obtain

$$\int_0^p \frac{dp}{\sqrt{(\alpha_1 - \alpha_2 p^2)(\beta_1 - \beta_2 p^2)}} = \int_0^t \frac{B-C}{A}\, dt. \tag{11.2.6}$$

Again putting $\sqrt{\frac{\alpha_2}{\alpha_1}}p = u$, we can deduce

$$\lambda\, t = \int_0^u \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \tag{11.2.7}$$

where $k^2 = \frac{\alpha_1\beta_2}{\beta_1\alpha_2}$, and $\lambda = \frac{\sqrt{\beta_1\alpha_2}\,(B-C)}{A}$.

We therefore derive the result in terms of Jacobi's elliptic function (see Dutta and Debnath (1965))

$$u = \operatorname{sn}(\lambda t, k). \tag{11.2.8}$$

As a matter of fact, we obtain p, q and r in terms of elliptic functions

$$p = \sqrt{\frac{\alpha_1}{\alpha_2}}\, \operatorname{sn}(\lambda t, k), \tag{11.2.9}$$

$$q = \sqrt{\alpha_1}\, \operatorname{cn}(\lambda t, k), \tag{11.2.10}$$

$$r = \sqrt{\beta_1}\, \operatorname{dn}(\lambda t, k). \tag{11.2.11}$$

An observational evidence revealed that the Earth does exhibit a phenomenon known as the *Eulerian nutation* of its axis. The Euler equations (11.2.1)–(11.2.3) are used to investigate this phenomenon. Although the Earth is not exactly a sphere, it has symmetry about its axis. Neglecting the moments of any external forces due to the Sun, the Moon and planets acting on the Earth, Euler's equation (11.2.1)–(11.2.3) with two equal principal moments ($A = B$) become

$$A\,\frac{dp}{dt} = (A-C)qr, \quad A\,\frac{dq}{dt} = (C-A)pr, \quad C\,\frac{dr}{dt} = 0, \tag{11.2.12}$$

where the z-axis is the axis of the Earth. It follows the last equation in (11.2.12) that

$$r = \omega_z = \text{constant} = n. \tag{11.2.13}$$

Writing $n(C - A)/A = \sigma$, the first and the second equations can be written as

$$\frac{dp}{dt} + \sigma q = 0 \quad \text{and} \quad \frac{dq}{dt} - \sigma p = 0, \tag{11.2.14}$$

which gives the equation

$$\ddot{p} + \sigma^2 p = 0. \tag{11.2.15}$$

This admits the solution

$$p = \omega_x = a\cos(\sigma t + \epsilon), \tag{11.2.16}$$

where a and ε are constants and

$$q = \omega_y = -\frac{1}{\sigma}\,\dot{p} = a\sin(\sigma t + \varepsilon). \tag{11.2.17}$$

This leads to $p^2 + q^2 = \omega_x^2 + \omega_y^2 = a^2$ so that the angular velocity vector $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ is of constant magnitude $\sqrt{a^2 + n^2}$ and it makes a constant angle $\theta = \tan^{-1}\left(\frac{a}{n}\right)$ with the Earth's axis. This vector $\boldsymbol{\omega}$, rotates around the Earth's axis with constant angular velocity σ. In other words, if the Earth's axis of rotation does not coincide with the geometrical axis, the former rotates round the latter. This phenomenon is known as the *Eulerian nutation*.

Euler first investigated one of the oldest eigenvalue problem concerning the buckling of a long homogeneous vertical column of uniform cross section and length ℓ subject to a compressive axial force or load P applied to its top. We recall equation (11.1.3) as

$$M(x) = EIy''. \tag{11.2.18}$$

When the column is deflected a small amount y from the vertical axis due to the constant load P, the bending moment is $M(x) = -Py$ so that equation (11.2.18) reduces to the form

$$EIy'' + Py = 0. \tag{11.2.19}$$

Since the column is assumed to be simply supported at the endpoints, there can be no displacement at these points. Thus, equation (11.2.19) is to be solved with the boundary conditions $y(0) = 0 = y(\ell)$. This constitutes an eigenvalue problem in $0 < x < \ell$. The solutions for the eigenvalue $\lambda = (P/EI)$ and the corresponding eigenfunctions are

$$\lambda = \lambda_n = (P_n/EI) = \frac{n^2\pi^2}{\ell^2} \quad \text{and} \quad y_n(x) = A_n \sin\left(\frac{n\pi x}{\ell}\right) \qquad (11.2.20\text{ab})$$

where $n = 1, 2, 3, \cdots$.

If P is sufficiently large, the vertical column would deflect or buckle. In other words, the column would buckle only when the applied load P is one of the values $P_n = (n^2\pi^2 EI)/\ell^2$ which are called the *critical Euler loads*. The smallest load $P_1 = (\pi^2 EI)/\ell^2$ that leads to buckling is called the *Euler critical load*, and the corresponding deflection, $y_1(x) = A_1 \sin\left(\frac{\pi x}{\ell}\right)$ is called the *first* (or *fundamental*) *buckling mode*. From a physical point of view, this result can be interpreted as follows. If $P < P_1$, the critical Euler load, the only solution is $y = 0$ and the column is undeflected. This situation is stable, that is, if the column is slightly bent, it will return to the original straight vertical form. When $P = P_1$, another solution also becomes possible in which the form of the column is $y_1(x)$ and this solution is stable.

In general, the differential equation governing the deflection of an elastic the column is given by

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2y}{dx^2}\right) + P\,\frac{d^2y}{dx^2} = 0, \qquad 0 < x < \ell. \qquad (11.2.21)$$

Or, equivalently,

$$y^{(iv)}(x) + k^2 y'' = 0, \qquad 0 < x < \ell, \qquad (11.2.22)$$

where $k^2 = (P/EI)$.

For a simply supported horizontal beam or vertical column, there is no displacement or bending moment at the endpoints. Mathematically, such conditions are prescribed as

$$y(0) = y''(0) = 0 = y(\ell) = y''(\ell). \qquad (11.2.23)$$

The general solution of the fourth order equation (11.2.22) is given by

$$y(x) = C_1 + C_2 x + C_3 \cos kx + C_4 \sin kx. \qquad (11.2.24)$$

Using the boundary conditions at $x = 0$ gives $C_1 = C_3 = 0$ and at $x = \ell$ leads to the pair of equations

$$C_2\,\ell + C_4 \sin k\ell = 0, \qquad k^2\, C_4\, \sin k\ell = 0. \qquad (11.2.25)$$

For nontrivial solutions to exist, we must choose $k = \left(\frac{n\pi}{\ell}\right)$ and set $C_2 = 0$. Hence, the eigenvalues and corresponding eigenfunctions are

$$k_n^2 = \frac{n^2\pi^2}{\ell^2}, \qquad y_n(x) = C_{4n} \sin\left(\frac{n\pi x}{\ell}\right), \qquad (11.2.26)$$

where C_{4n} are constants and $n = 1, 2, 3, \cdots$.

For these admissible values of k_n^2, the critical buckling loads are

$$P_n = \frac{n^2\pi^2 EI}{\ell^2}, \qquad n = 1, 2, 3, \cdots . \tag{11.2.27}$$

The largest load that the beam can withstand before buckling is the *Euler load*

$$P_1 = \frac{\pi^2 EI}{\ell^2}, \tag{11.2.28}$$

corresponding to the *fundamental buckling mode*

$$y_1(x) = C_{41} \sin\left(\frac{\pi x}{\ell}\right). \tag{11.2.29}$$

In the case of an inhomogeneous column with the ends fixed by hinges so that the flexural rigidity varies along the column, that is, $EI = \phi(x)$, where $\phi(x)$ is a given positive function. Consequently, equation (11.2.19) becomes

$$y'' + \frac{P}{\phi(x)}\, y = 0. \tag{11.2.30}$$

Putting $(y'/y) = u$ so that $y = \exp\left[\int u\, dx\right]$, equation (11.2.30) reduces to the first order nonlinear equation

$$\frac{du}{dx} + u^2 + \frac{P}{\phi(x)} = 0. \tag{11.2.31}$$

Euler worked out an example with $\phi(x) = \kappa_0(\alpha + \beta x)^\lambda$ where $\lambda = 4$, and obtained the exact solution

$$y(x) = A(\alpha\ell + \beta x) \sin\left[\frac{\gamma \ell x}{\alpha(\alpha\ell + \beta x)}\right], \tag{11.2.32}$$

where $\gamma = \sqrt{P_{cr}/\kappa_0}$ so that the critical load is

$$P_{cr} = \kappa_0 \left(\frac{\pi}{\ell}\right)^2 \alpha^2(\alpha + \beta)^2. \tag{11.2.33}$$

Subsequently, Euler extended this work to study the elastic stability of conical columns, and then the stability of an inhomogeneous elastic rod with arbitrary degree of inhomogeneity λ. All these reveal that Euler made stability analysis of various geometrical configurations with the critical parameters determining loss of stability.

As early as 1739, Euler investigated the phenomenon of *resonance* for a sinusoidal oscillation of simple harmonic motion. In his later work, he expanded the concept of resonance in the theory of forced vibrations. He gave an explanation of the phenomenon of resonance in his article "On a new type of oscillations", where he investigated the problem of forced linear oscillations of a simple harmonic oscillator under the action of a harmonic load and obtained a solution in an integral form. In a special case where the frequency of the forcing function is equal to the natural frequency of the harmonic oscillator, his solution predicted an unbounded growth of the oscillations. Among many different models exhibiting resonance phenomenon, we consider an example of the transverse vibrations of an elastic beam of uniform cross section and length ℓ so that the end $x = 0$ is hinged, that is, the displacement is zero at $x = 0$, and the end $x = \ell$ is hinged on a support which moves parallel to y-axis in a simple harmonic manner. With no external force acting along the beam, the displacement function $y(x,t)$ satisfies the partial differential equation

$$m\,\frac{\partial^2 y}{\partial t^2} + EI\,\frac{\partial^4 y}{\partial x^4} = 0, \qquad 0 < x < \ell, \quad t > 0, \tag{11.2.34}$$

where m is the mass of the beam per unit length. If the beam is initially at rest along the x-axis, the appropriate initial and boundary conditions of this problem are

$$y(x,0) = 0 = y_t(x,0) \quad 0 < x < \ell, \tag{11.2.35}$$

$$y(0,t) = 0 = y_{xx}(0,t) \quad t > 0, \tag{11.2.36}$$

$$y(\ell,t) = A\sin\omega t, \quad y_{xx}(\ell,t) = 0, \quad t > 0, \tag{11.2.37}$$

where A is a constant and ω is the frequency of the support.

Making reference to Debnath and Bhatta (2007), the Laplace transform $\bar{y}(x,s)$ of $y(x,t)$ satisfies the following boundary problem

$$a^2\,\frac{d^4\bar{y}}{dx^4} + s^2\,\bar{y} = 0, \qquad 0 < x < \ell, \tag{11.2.38}$$

$$\bar{y}(0,s) = \bar{y}_{xx}(0,s) = \bar{y}_{xx}(\ell,s) = 0, \quad \bar{y}(\ell,s) = \frac{A\omega}{s^2+\omega^2}, \tag{11.2.39}$$

where $a^2 = (EI/m)$.

It is convenient to write $s = iap^2$ so that $s^2 = -a^2p^4$. Then the general solution of (11.2.38) is given by

$$\bar{y}(x,s) = c_1\sin px + c_2\cos px + c_3\sinh px + c_4\cosh px, \tag{11.2.40}$$

where the constants $c's$ can be functions of the parameter s, and are determined by (11.2.39). The solution (11.2.40) becomes

$$\bar{y}(x,s) = \frac{A\omega}{s^2+\omega^2}\cdot\frac{\sin px\sinh p\ell + \sinh px\sin p\ell}{2\sin p\ell\sinh p\ell}. \tag{11.2.41}$$

This solution is the analytic function of s except at points where the denominator vanishes. The point $s = 0$ is a removable singularity, and the singular points of $\bar{y}(x, s)$ are

$$s = \pm i\omega, \quad s = \pm \frac{in^2\pi^2 a}{c^2} = \pm\, i\,\omega_n, \quad n = 1, 2, \cdots. \tag{11.2.42}$$

If ω is different from all the numbers ω_n, the derivative of the denominator in (11.2.41) does not vanish at any of the points (11.2.42), nor does the numerator except for particular values of x. Those points are therefore simple poles of $\bar{y}(x, s)$. Using the theory of residues at the simple poles, the displacement function has the form

$$y(x,t) = a_0(x)\cos\left[\omega t + \theta_0(x)\right] + \sum_{n=1}^{\infty} a_n(x)\cos\left[\omega_n t + \theta_n(x)\right], \tag{11.2.43}$$

where $a_0(x)$ and $a_n(x)$ are known.

If ω is equal to one of the numbers $\omega_n = \left(n^2\pi^2 a\right)/\ell^2$, then $\bar{y}(x, s)$ has a double pole at the points $s = \pm\, i\omega_n$, $y(x, t)$ contains terms of the type

$$t\, a_n(x)\cos\left[\omega_n t + \theta_n(x)\right], \qquad a_n(x) \neq 0 \tag{11.2.44}$$

which tends to infinity as $t \to \infty$, and hence, represents an unstable solution. This unstable oscillatory solution is called *resonance*, and ω_n are called the *resonant frequencies*.

In case the hinge at $x = \ell$ is kept fixed and a simple harmonic bending moment acts on that end of the beam, the conditions at $x = \ell$ have the form

$$y(\ell, t) = 0, \qquad y_{xx}(\ell, t) = A\sin\omega t. \tag{11.2.45}$$

In this case, the solution $\bar{y}(x, s)$ then has the same denominator as it does in equation (11.2.41). So, resonance occurs at the resonant frequencies ω_n as given above.

Another resonance model arises when both ends of a beam of length ℓ are built into rigid supports. The beam is initially at rest with no displacements. A simple harmonic force per unit length acts along the entire span in a direction normal to the beam so that the transverse displacement satisfies the equation

$$\frac{\partial^2 y}{\partial t^2} + a^2\,\frac{\partial^4 y}{\partial x^4} = A\sin\omega t, \tag{11.2.46}$$

where $a^2 = (EI/m)$. In this case, resonance occurs when $\omega = \left(4a\,\alpha_n^2/\ell\right)$, where α_n is root of the transcendental equation $\tan\alpha = -\tanh\alpha$.

If a compressive load is applied impulsively to the ends of an elastic beam or rod, then the shape at which instability occurs differs from that obtained in cases static loading. The main difference arises in the situation that under dynamic loading the form of the buckling develops at a high frequency. In other words, the loss of stability in a beam occurs in a harmonic range higher than in the statical Eularian problem. Thus, many problems concerning the stability of elastic beams or rods under impulsive loading *cannot* be explained in terms of the statical analysis of stability. It is then necessary to use the dynamic equation of small deformations of the beam in the form

$$m\,\frac{\partial^2 y}{\partial t^2} + EI\,\frac{\partial^4 y}{\partial x^4} + P\,\frac{\partial^2 y}{\partial x^2} = f(x), \tag{11.2.47}$$

where $f(x)$ is the initial deformation. Such problem is solved with the boundary conditions

$$y(x,t) = y_{xx}(x,t) = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = \ell. \tag{11.2.48}$$

The appropriate solution of this problem is assumed in the form of a Fourier sine series

$$y(x,t) = \sum_{n=1}^{\infty} Y(t)\sin\left(\frac{n\pi x}{\ell}\right). \tag{11.2.49}$$

If the initial deformation function f is also expanded in a Fourier sine series as

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{\ell}\right), \tag{11.2.50}$$

where B_n are the Fourier coefficient substituting (11.2.49) and (11.2.50) in equation (11.2.47) gives

$$m\,\frac{d^2 Y_n}{dt^2} + \frac{\pi^4 EI}{\ell^4} n^2\left(n^2 - \frac{P}{P_1}\right) Y_n = B_n, \tag{11.2.51}$$

where $P_1 = (\pi^2 EI)/\ell^2$ is the critical Euler load. Equation (11.2.51) admits sinusoidal solution provided $n^2 > (P/P_1)$ and exponential solution if $n^2 < (P/P_1)$ leading to instability as $t \to \infty$.

Many other related statical or dynamical problems of the Euler–Bernoulli equation for the vertical deflection of an infinite elastic beam on an elastic foundation under the action of prescribed load with or without damping have also been investigated by many authors including Stadler and Shreeves (1972), Sheehan and Debnath (1972) and Debnath and Bhatta (2007).

Based on the classical work of Euler, considerable new progress has been made on further extension and development of the analysis of vibrations to elastic systems under periodic forces. At the same time, new questions concerned with oscillations and stability of elastic systems subject to periodic forces and the related topic of parametric resonance have also been investigated in great details. Many new and interesting results dealing with elastic viscoelastic, and hydroelastic properties emerging in problems of parametric resonance. On the other hand, approximate solutions of the inherently nonlinear problem of elastic stability and post-buckling behavior have been discussed by many authors including Friedrichs and Stoker (1941) and Koiter (1943). A more general analysis which is rigorous in an asymptotic sense for the initial state of post-buckling behavior has been developed in 1940s. It has been shown that the initial state of post-buckling behavior is determined completely by the stability or instability of equilibrium at the critical load itself. In other words, the initial state of the post-buckling behavior is governed by the answer to the question whether the critical bifurcation point still belongs to the stable part of the fundamental branch of equilibrium or to the unstable part. This equation can only be answered by the study of higher order variations of the potential energy.

In the twentieth century, several authors including Nikolai (1939), Dinnik (1950), Ishlinkskii (1954), and Panovko and Gubanova (1964) made stability and bifurcation analyses of columns, beams, elastic and viscoelastic bodies. Euler's remarkable work on the stability analysis of elastic systems has been developed much further in the nineteenth and twentieth centuries because of the tremendous need for solving new physical and engineering problems. Many subsequent attempts have been made to apply Euler's analytical methods to problems of elastic stability in both conservative and non-conservative systems. Among others, Bolotin's (1961) book on *Nonconservative problems in the theory of elasticity* is a good example of recent developments of the theory elastic stability which brings together within the framework of a very general approach to many nonconservative dynamical problems including the problems of aeroelasticity and problems of instability in high-speed rotors.

11.3 Euler's Research on Elastic Curves

The discovery of Robert Hooke's (1635-1703) famous law of proportionality of stress and strain in 1660, and the formulation of the general equations of equilibrium and formulation of motion by Louis M. H. Navier in 1821 are the two great landmarks in the history of elasticity. In 1705, James Bernoulli first investigated the elastic curve (or *elastica*) produced by the resistance of a bent rod from the extension and contraction of its longitudinal filaments.

In 1727, while he was in Basel, Euler began his research on oscillations of an elastic ring. He then considered elastic curves (or *elastica*) in his paper entitled 'Solution of the problem of finding the curve formed by an elastic band loaded by arbitrary forces at each of its points' published in 1728. In this work, he provided an unified treatment of several elastic curves and also derived a differential equation which described the elastica including the catenary, the velaria, and the linteria. His further research on elastic curves was strongly motivated by a series of correspondence with Daniel Bernoulli during 1738-1744. In particular, in his letter to Daniel on October 20, 1742, Euler suggested the use of a variational principle for the study of elastic curves. He formulated the problem as follows: "Among all curves of prescribed length that not only pass through points A and B, but also tangential at those points to prescribed straight lines [through A and B], determine that one for which the expression $\int \frac{ds}{R^2}$ is least," where ds is the arclength and R is radius of curvature of the curve. Euler's major research in elasticity was published in an appendix, *De Curvis Elastica*, to his 1744 monumental treatise on the Calculus of Variations entitled *Methodus inveniendi lineas curvas maximi minimive proprietate guardentes, ...* (*A method for finding curves with maximal or minimal properties, ...*). In this appendix, he announced a mathematical analysis of elastic curves and also acknowledged the outstanding contributions of Daniel Bernoulli in the introduction to his treatise. Euler's appendix *De Curvis Elastica* was translated into English by Oldfather et al. (1933) under the title "Leonhard Euler's Elastic Curves". It is worthy to note that Euler began his appendix with a discussion of some metaphysical principles, and then described the universal applicability of the principles of maxima and minima as follows: "For since the fabric of the Universe is most perfect and the work of a most wise Creator, nothing at all takes place in the Universe in which some rule of maxima or minima does not appear."

In his *De Curvis Elastica*, Euler gave mathematical analysis of many problems of elasticity dealing with deflection of a thin elastic rod under a terminal load, a complete classification of *nine* distinct equilibrium forms of elastic curves, curvature of elastica and oscillations of elastica in many different geometrical configurations. This work led Euler to introduce the fundamental idea of elastic stability of linear and nonlinear problems, and to discover the resonance phenomenon and bifurcation analysis. His major work on dynamics of elastic bodies was concerned with the study of vibrating strings, rods, membranes, columns and plates. He made many invaluable contributions to major problems of elasticity, resistance of materials, and structural mechanics.

(i) The Euler Problem of Elastica

The problem is to determine the shape of a thin rod, straight and prismatic in the unstressed state, under the action of forces and couples applied at its end points only. When the rod is bent in a principal plane under the action of a force F applied at the end from which s is measured, the central line becomes a plane curve and there is no twist. This corresponds to the Kirchhoff kinetic analogue which is a rigid pendulum of weight F, turning about the fixed horizontal x-axis. The motion of the pendulum is governed by the energy equation and the initial data. We assume that $\theta = \theta(s)$ is the angle which the tangent of the central line at any point makes with the line of action of the applied force F, and drawn in the sense of increasing s. The shape of the curve is called *elastica* and determined by the equilibrium equation

$$\frac{d^2\theta}{ds^2} + \lambda \sin\theta = 0, \tag{11.3.1}$$

where $\lambda = \frac{F}{B}$, and B is the flexural rigidity of the plane of bending. The shape of elastica is different depending on whether there are or are not inflexions.

The first integral of (11.3.1) is given by

$$\frac{1}{2}\left(\frac{d\theta}{ds}\right)^2 - \lambda\cos\theta = C, \tag{11.3.2}$$

where C is an arbitrary constant.

At an inflexion, $\frac{d\theta}{ds} = 0$ and the flexural couple vanishes so that the rod can be the shape of inflexional elastica by terminal force alone, without couple. The kinetic analogue of the inflexional elastica is an *oscillating pendulum*. On the other hand, when the rod takes the form of a non-inflexional elastica, both terminal forces and couples are required. The corresponding kinetic analogue is a *revolving pendulum* when there are no terminal forces, the rod is bent into an arc of a circle, and the kinetic analogue is a rigid body revolving about the horizontal axis that passes through its center of gravity.

We consider two forms of the elastica: (a) inflexional or (b) non-inflexional elastica according as there are or are not inflexions.

(a) Inflexional Elastica

If s is measured from an inflexion and $\alpha = \theta(0)$, equation (11.3.2) assumes the form

$$\left(\frac{d\theta}{ds}\right)^2 = 2\lambda\,(\cos\theta - \cos\alpha) \tag{11.3.3}$$

or equivalently

$$\left(\frac{d\theta}{ds}\right)^2 = 4\lambda\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right). \tag{11.3.4}$$

This gives the solution

$$2\sqrt{\lambda}\,s = \int_0^{\theta}\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)^{-\frac{1}{2}} d\theta. \tag{11.3.5}$$

Substituting $\sin\frac{\theta}{2} = uk$, $k = \sin\frac{\alpha}{2}$, we obtain a standard elliptic integral (see Dutta and Debnath (1965)) in the form

$$2\sqrt{\lambda}\,s = \int_0^{u}\frac{du}{[(1-u^2)(1-k^2u^2)]^{\frac{1}{2}}}. \tag{11.3.6}$$

This gives the solution in terms of the Jacobi sn function with modulus k:

$$u(s,k) = sn(\sqrt{\lambda}\,s, k). \tag{11.3.7}$$

In fact, this represents the exact solution of the nonlinear problem of the inflexional elastica.

In order to determine the shape of the elastica, we take (x, y) as the coordinate of a point to fixed axis so that the line of thrust coincides with the x-axis. In view of the fact that $(dx, dy) = (\cos\theta, \sin\theta)\,ds$, the solutions are given by

$$x = 2\sqrt{\lambda}\left[E\left(\sqrt{\lambda}\,s, k\right) - 1\right] - s, \qquad y = 2k\sqrt{\lambda}\left[1 - cn\left(\sqrt{\lambda}\,s, k\right)\right], \tag{11.3.8ab}$$

where $E(z,k)$ is the Legendre elliptic integral of the second kind, $cn(z,k)$ is the Jacobi elliptic function; and s is connected by the functional relation (11.3.7).

The inflexions are determined by $\cos\theta = \cos\alpha$ or $cn^2(\sqrt{\lambda}\,s, k) = 0$. This means that the arc between two consecutive inflexions is $(2K/\sqrt{\lambda})$ where $4K$ is the real period of $cn(z)$ and $sn(z)$. These inflexions are equally spaced along the x-axis at intervals

$$\frac{2}{\sqrt{\lambda}}[2E(K) - K], \tag{11.3.9}$$

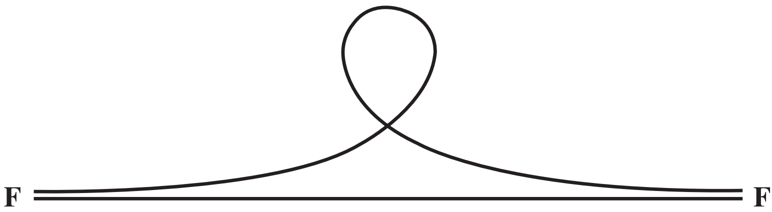


Fig. 11.1 A single loop.

where $E(K) \equiv E(K, k)$ is the complete elliptic integral of the second kind.

The points at which the tangents are parallel to the line of thrust are determined by $\sin\theta = 0$ and $sn(\sqrt{\lambda}\, s)dn(\sqrt{\lambda}\, s) = 0$. This implies that s is an even multiple of K. It turns out that the inflexion elastica consists of a series of bays which are separated by points of inflexion and divided into equal half-bays by the points at which the tangents are parallel to the line of thrust. Finally, it follows that the curve changes its form as α increases and all *eight* different forms are shown in Love's book (1944). When $\alpha > \frac{\pi}{2}$, x is negative for all small values of $s\sqrt{\lambda}$, and when $\alpha = \pi$, the curve forms a single loop as shown in Figure 11.1. In this limiting case ($\alpha = \pi$), the rod of infinite length forms a single loop. The pendulum begins to close to the position of unstable equilibrium and just makes one complete revolution.

(b) Non-Inflexional Elastica

In this case, equation (11.3.3) can be written in the form

$$\left(\frac{d\theta}{ds}\right)^2 = 2\lambda\left[\cos\theta + 1 + \frac{2(1-k^2)}{k^2}\right], \qquad k < 1. \tag{11.3.10}$$

This can be solved exactly in terms of the argument of the Jacobi elliptic function $sn(u, k) = \sin\frac{\theta}{2}$, where $u = \frac{\sqrt{\lambda}}{k}\, s$. Measuring s at a point when $\theta = 0$, we obtain

$$\frac{d\theta}{ds} = \frac{2}{k}\sqrt{\lambda}\; dn\, u, \qquad sn\, u = \sin\frac{\theta}{2}. \tag{11.3.11}$$

Thus, the coordinates x and y are given by

$$x = k\sqrt{\lambda}\left[\left(1 - \frac{2}{k^2}\right)u + \frac{2}{k^2}\, E(u, k)\right], \quad y = -\frac{2\sqrt{\lambda}}{k} dn(u, k). \tag{11.3.12ab}$$

It follows from a careful analysis that the curve forms a series of loops lying altogether one side of the x-axis as shown in Figure 11.2.

We next include the theory of buckling of long thin strut under thrust developed by Euler, and discussed by Love (1944) in his great book. We

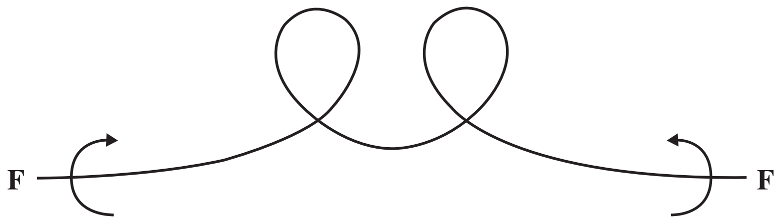


Fig. 11.2 A series of loops.

follow Love's discussion by adding the limiting form of the elastica as $\alpha \to 0$ and $\theta \to 0$. In the limit as $\theta \to 0$, we replace $\sin\theta$ by θ so that the solutions of the linearized equation (11.3.1) become

$$\theta = \alpha\cos(\sqrt{\lambda}\, s), \qquad x = s, \qquad y = \alpha\sqrt{\lambda}\,\sin(x\sqrt{\lambda}). \tag{11.3.13}$$

Thus, the elastic curve is approximately a sine-curve of small amplitude α. The distances between two consecutive inflexions is $(\pi/\sqrt{\lambda})$. Thus, a long straight elastic rod can be bent by applying forces at its end in a direction parallel to that of the rod when undeflected provided the length ℓ and force F satisfy the inequality

$$\lambda\,\ell^2 > \pi^2 \qquad \text{or} \qquad \ell^2 F > \pi^2 B. \tag{11.3.14}$$

If the direction of the rod at the end is the same as that of the applied forces, the length is half that between consecutive inflexions and inequality (11.3.14) becomes $4\lambda\ell^2 > \pi^2$ or $4\ell^2 F > \pi^2 B$. On the other hand, if the ends of the rod are subjected to remain in the same line, the length is twice that between consecutive inflexions, and then inequality (11.3.14) becomes $\lambda\ell^2/4\pi^2$ or $\ell^2 F > 4\pi^2 B$. It follows from this analysis that if the length is slightly larger than $\left(\frac{1}{2}\,\frac{\pi}{\sqrt{\lambda}}\right)$, or the applied force F is slightly greater than $\left(\frac{1}{4}\,\pi^2 B/\ell^2\right)$, the rod is deflected under this load, so that the central line assumes the form of one half-bay of a sine curve of small amplitude. If the length of the rod is less than the critical value, it simply contract under the load. On the other hand, if the length is greater than the critical length, the equilibrium of the rod is *unstable.*

(ii) The Euler Bifurcation Problem of Elastica

We next present a bifurcation analysis of the Euler problem in some detail. We assume that the axis of the elastic column of length ℓ coincides with the x-axis. It is also assumed that the column remains straight before the application of the compressive thrust at its end. We denote the angle between the tangent to the column's axis and the x-axis by $\theta(x)$. The horizontal and vertical displacements of the buckled axis are denoted by $u(x)$ and $w(x)$ respectively. We use the length scale ℓ to normalize the problem so that $0 \leq x \leq 1$.

The nonlinear boundary value problem is then governed by

$$\theta'' + \lambda \sin\theta = 0, \qquad 0 \leq x \leq 1, \tag{11.3.15}$$

$$\theta'(0) = \theta'(1) = 0, \tag{11.3.16}$$

$$u' = \cos\theta - 1, \quad u(0) = 0, \qquad 0 \leq x \leq 1, \tag{11.3.17}$$

$$w' = \sin\theta, \quad w(0) = w(1) = 0, \qquad 0 \leq x \leq 1, \tag{11.3.18}$$

where the prime denotes the differentiation with respect to x, and λ is a non-negative parameter proportional to the applied thrust.

The linearized problem (11.3.15)–(11.3.18) admits eigenfunction solutions

$$\theta_n = A_n \sin n\pi x, \qquad n = 0, 1, 2, 3, \cdots, \tag{11.3.19}$$

corresponding to the eigenvalues

$$\lambda_n = n^2\pi^2, \qquad n = 0, 1, 2, 3, \cdots. \tag{11.3.20}$$

On the other hand, it can be proved that the nonlinear problem (11.3.15)–(11.3.18) also admits non-trivial solutions. In fact, it is possible to determine an explicit expression for the curves which bifurcate from eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$. In order to find these curves, we write (11.3.15) in the form

$$\left(\frac{d\theta}{dx}\right)^2 = 2\lambda\left(\cos\theta - \cos\alpha\right), \qquad \theta(0) = 0. \tag{11.3.21}$$

We introduce a change of variable ϕ defined by

$$\sin\frac{\theta}{2} = k\sin\phi, \qquad k = \sin\frac{\alpha}{2}, \tag{11.3.22}$$

and transforms (11.3.15)–(11.3.18) into the form

$$\phi' = \mu\left(1 - k^2\sin^2\phi\right)^{1/2} \tag{11.3.23}$$

$$\phi(0) = \left(2m + \frac{1}{2}\right)\pi, \qquad m = 0, \pm 1, \pm 2, \cdots \tag{11.3.24}$$

$$\phi(1) = \left(n + \frac{1}{2}\right)\pi, \qquad n = \pm 1, \pm 2, \cdots. \tag{11.3.25}$$

This differential system admits the solution

$$\mu x = \int_{\phi(0)}^{\phi(x)} \frac{d\theta}{\left(1 - k^2\sin^2\theta\right)^{1/2}}, \tag{11.3.26}$$

which gives the value of μ at $x = 1$

$$\mu = \int_{\phi(0)}^{\phi(1)} \frac{d\theta}{\left(1 - k^2 \sin^2\theta\right)^{1/2}}. \tag{11.3.27}$$

Integral (11.3.26) or (11.3.27) is somewhat similar to that of the complete elliptic integral of the first kind defined by Dutta and Debnath (1965) in the form

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\left(1 - k^2 \sin^2\theta\right)^{1/2}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right), \tag{11.3.28}$$

where $F(a, b, c, x)$ is the hypergeometric function. The function $K(k)$ satisfies the properties: (i) $K(0) = \frac{\pi}{2}$, (ii) $\frac{dK}{dk} >$ or < 0 according as $k > 0$, or < 0, (iii) $K(k) \to \infty$ as $k \to \pm 1$ and (iv) for $k \ll 1$ (11.3.28) implies that

$$K(k) = \int_0^{\pi/2} \left(1 + \frac{1}{2} k^2 \sin^2\theta\right) d\theta = \frac{\pi}{2}\left(1 + \frac{k^2}{4}\right). \tag{11.3.29}$$

In view of the fact that the limits of integration in (11.3.27) is a multiple of π, it follows from (11.3.27) that μ can be expressed as

$$\mu = \mu_n(k) = 2nK(k), \qquad \mu_n(0) = n\pi = \lambda_n, \qquad n = 1, 2, 3, \cdots. \tag{11.3.30}$$

Furthermore, for $k \ll 1$,

$$\mu_n(k) \sim n\pi\left(1 + \frac{k^2}{4}\right). \tag{11.3.31}$$

Thus, in the neighborhood of $k = 0$, $\mu_n(k)$ is approximately a parabola.

The above properties of $K(k)$ enables us to draw the Figure 11.3. Thus, when $\lambda = 0$, the only solutions are $\theta = \theta_0 =$ constant which correspond to the equilibrium position of the rod. In the case $\lambda \neq 0$, the complete solution of the problem is described above. Indeed, the buckled state branching from λ_n exists for all $\lambda \geq \lambda_n$ which can be seen from Figure 11.3. Thus, the solutions of the Euler elastica problem confirm one of the main features of bifurcation, namely, the sharp transition in solution multiplicity occurs at the bifurcation points. As shown in Figure 11.3, the bifurcation branches exist near the bifurcation points λ_n only to the right of the bifurcation points. This corresponds to the supercritical bifurcation.

It is clear from the above discussion that Euler's major work on elasticity has provided a continuing strong influence on many modern areas of research such as the theory cantilever beam and the theory of flexure of beams of finite section due to Charles A. Coulomb (1736-1805) in 1773.

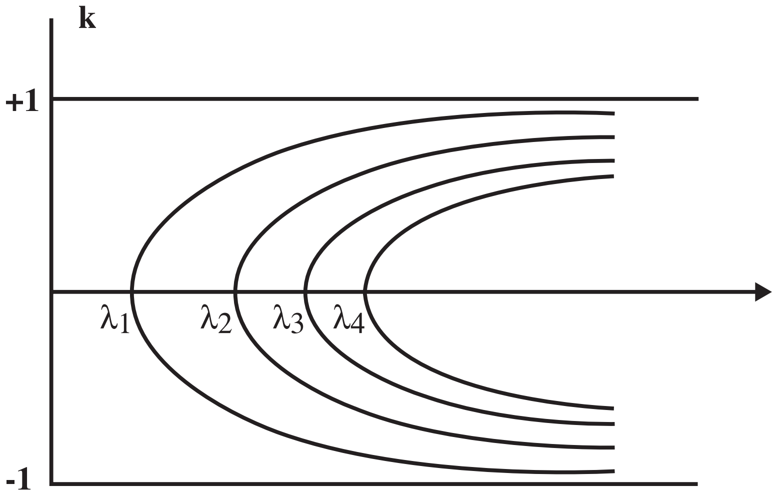


Fig. 11.3 Supercritical bifurcation curves.

Euler's research on elasticity has also provided great impact on later work on the elastica by Gustav Kirchhoff (1824-1887) in 1859, Max Born (1882-1970) in 1906 and Bryant and Griffiths (1986), and applications to the size and shapes of biological organisms due to Greenhill (1881), and the elastic properties of the DNA molecule due to Benham (1977 and 1979).

11.4 Impact of Euler's Work on Modern Aerodynamics

In the study of aircraft dynamics, it is natural to use a coordinate system fixed to the Earth to describe the position and orientation of an aircraft relative to the Earth. The position relative to the Earth is usually described in terms of latitude, longitude, and elevation above the mean sea level. Since the coordinate system is not a Cartesian system, it usually introduces some problems in the formulation of aircraft dynamics. However, it is convenient to describe aircraft position and orientation in terms of an *Earth-fixed coordinate system* (x_f, y_f, z_f), that can be considered an *inertial coordinate system.* In this Earth-fixed coordinate system, the components of the inertia tensor in the equations of motion become time dependent which introduces mathematical problems. This problem can be simplified by formulating the angular momentum equations in terms of a coordinate system that is fixed to the aircraft with the origin located at the aircraft's center of gravity. This is called the *non-inertial* or the *body-fixed coordinate system*, (x_b, y_b, z_b). On the other hand, the aerodynamic forces and moments acting on an aircraft depend *not* on the velocity of the aircraft relative to the ground, but rather, on the velocity relative to the surrounding air. Hence, the surrounding air can produce wing motion relative to the Earth. In other words, the atmosphere in the immediate vicinity of the aircraft moves at a uniform velocity relative to the Earth. Thus, under this approximation, the *atmosphere-fixed coordinate system*, (x_a, y_a, z_a) can also be regarded as an inertial coordinate system. These three different coordinate systems are usually used in formulating the aircraft equations of motion. In summary, the position and orientation of the aircraft are best described in terms of the Earth-fixed coordinate systems, the components of the inertia tensor are most easily described in terms of a body-fixed coordinate system, and forces and moments are described in terms of the atmosphere-fixed coordinate system. However, the atmosphere-fixed coordinate system has relatively less impact on the equations of motion, and the final equations of motion are completely expressed in terms of the Earth-fixed and body-fixed coordinates.

The orientation of an aircraft relative to the Earth can be described in terms of what are known as the *Euler angles*. Thus, it is necessary to derive the transformation between the two coordinate systems (x_f, y_f, z_f) and (x_b, y_b, z_b) in terms of the three Euler-angle rotations. We first treat each of the three rotations as a separate transformation and then combine the three transformation equations into one transformation equation. We denote a velocity vector $\mathbf{v}$ with components $(v_{x_f}, v_{y_f}, v_{z_f})$ in the inertial coordinate system (x_f, y_f, z_f) and the components $(v_{x_b}, v_{y_b}, v_{z_b})$ in the non-inertial (x_b, y_b, z_b) coordinate system. Making reference to standard texts such as Phillips (2004), we write a single transformation equation in the matrix form

$$\begin{bmatrix} v_{x_f} \\ v_{y_f} \\ v_{z_f} \end{bmatrix} = A \begin{bmatrix} v_{x_b} \\ v_{y_b} \\ v_{z_b} \end{bmatrix}, \tag{11.4.1}$$

where A is a 3×3 matrix of the form

$$A = \begin{bmatrix} C_\theta C_\psi & S_\phi S_\theta C_\psi - C_\phi S_\psi & C_\phi S_\theta C_\psi + S_\phi S_\psi \\ C_\theta S_\psi & S_\phi S_\theta S_\psi + C_\phi C_\psi & C_\phi S_\theta S_\psi - S_\phi C_\psi \\ -S_\theta & S_\phi C_\theta & C_\phi C_\theta \end{bmatrix}, \tag{11.4.2}$$

$S_\phi = \sin\phi$, $C_\phi = \cos\phi$, $S_\theta = \sin\theta$, $C_\theta = \cos\theta$, $S_\psi = \sin\psi$, $C_\psi = \cos\psi$ and

ϕ, θ, ψ are the three Euler angles representing the bank angle, the elevation angle and the azimuth angle respectively.

The inverse transformation equation is then given by

$$\begin{bmatrix} v_{x_b} \\ v_{y_b} \\ v_{z_b} \end{bmatrix} = A^T \begin{bmatrix} v_{x_f} \\ v_{y_f} \\ v_{z_f} \end{bmatrix}, \tag{11.4.3}$$

where A^T is the transpose of the matrix A.

In the body-fixed coordinates, the velocity of the aircraft relative to the surrounding air is (u, v, w) and in the Earth-fixed coordinates, the velocity of the aircraft relative to the Earth is $(\dot{x}_f, \dot{y}_f, \dot{z}_f)$ which is the time rate of change of the position vector (x_f, y_f, z_f). The velocity of the aircraft is the sum of the ground speed and the wind velocity so that

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_f \\ \dot{x}_f \end{bmatrix} = A \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}, \tag{11.4.4}$$

where (w_x, w_y, w_z) is the wind velocity (or the velocity of the atmosphere relative to the Earth).

Using these same Euler angle definitions, we obtain the relation between the time rate of change of the Euler angle and the body-fixed angular velocity vector (p, q, r) in the form

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & S_\phi S_\theta / C_\theta & C_\phi S_\theta / C_\theta \\ 0 & S_\phi C_\theta & -S_\psi \\ 0 & S_\phi / C_\theta & C_\phi / C_\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{11.4.5}$$

In terms of this particular set of Euler angles, equations (11.4.4) and (11.4.5) represent the *six kinematic transformation equations* which are used to update the position and orientation of the aircraft with time. It is noted that the transformation equations (11.4.5) contains a singularity which can be seen in the first and third equations of (11.4.5). Evidently, these four terms become singular when $\cos\theta = 0$, that is, $\theta = \pm\frac{\pi}{2}$. At these points the transformation breaks down, and integration of the Euler angles becomes indeterminate. This singularity in the integration of the Euler angles is usually known as the *gimbal lock* which physically occurs when the nose of the aircraft is pointed straight up or straight down. This singularity is really problematic in aircraft flight simulation. Despite the singularity, the Euler angle formulation is widely used because the three Euler angles have simple physical interpretations. In summary, the six degrees of freedom rigid body equations of motion for an aircraft in flight are formulated in terms of Euler angles. It is convenient to solve the linearized version of the Euler angle formulation and then use the solution to discuss several aspects of aircraft dynamics. However, there are many cases where the linearized equations of motion are no longer valid. For those cases, it is necessary to study the coupled nonlinear formulation, but it is almost impossible to handle such nonlinear problems without further assumptions and approximations. Due to the existence of the singularity, equations (11.4.4)–(11.4.5) are not normally employed to determine the Euler angles. Instead, other methods including the *direction cosine transformation* or the *quaternion transformation* are normally employed. These methods contain no singularity and are frequently used by aircraft engineers.

We next briefly describe the direction cosine formulation by introducing the direction cosine matrix that can be obtained from any symmetric or asymmetric Euler angle sets. In order to avoid the singularities involved in the Euler angle formulation, we use the matrix A in (11.4.2) as a direction cosine matrix and treat the nine elements of this matrix as a fundamental description of orientation. The elements of this matrix are called the *direction cosines.* If these nine direction cosines are known, the components of any arbitrary vector in the body fixed coordinate system can simply be related to the nine components of the same vector in the inertial coordinate system through the definition of the direction cosine matrix $[C_{ij}]$ by

$$\begin{bmatrix} v_{x_b} \\ v_{y_b} \\ v_{z_b} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} v_{x_f} \\ v_{y_f} \\ v_{z_f} \end{bmatrix}. \tag{11.4.6}$$

Since the inverse of the matrix $[C_{ij}]$ is equal to the transpose matrix $[C_{ji}]$, the velocity of the aircraft can be expressed as the sum of the ground speed and the wind velocity so that

$$\begin{bmatrix} \dot{x}_f \\ \dot{x}_f \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}. \tag{11.4.7}$$

In order to write the kinematic equations associated with the direction cosine formulation, we obtain a set of differential equations relating the time derivatives of the nine elements of the direction cosine matrix $[C_{ij}]$ to the body fixed angular velocity vector in the matrix form

$$\begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} & \dot{C}_{13} \\ \dot{C}_{21} & \dot{C}_{22} & \dot{C}_{23} \\ \dot{C}_{31} & \dot{C}_{32} & \dot{C}_{33} \end{bmatrix} = \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}. \tag{11.4.8}$$

These nine differential equations are called the *Poisson kinematic equations*.

The orientation of an aircraft has only three degrees of freedom. But there are nine elements, C_{ij} in the direction - cosine matrix. So, these nine elements cannot be independent. There must be six degrees of redundancy in the direction cosine formulation. To eliminate these six degrees of freedom, some constraints are required. We use the fact that rigid body rotation is an orthogonal transformation so that the direction cosine matrix $[C_{ij}]$ must be orthogonal. This means that the inverse of $[C_{ij}]$ is equal to its transpose which leads to the six constraints as

$$C_{11}^2 + C_{21}^2 + C_{31}^2 = C_{12}^2 + C_{22}^2 + C_{32}^2 = C_{13}^2 + C_{23}^2 + C_{33}^2 = 1, \tag{11.4.9}$$

$$C_{11}C_{12} + C_{21}C_{22} + C_{31}C_{32} = 0, \quad C_{11}C_{13} + C_{21}C_{23} + C_{31}C_{33} = 0,$$
$$C_{12}C_{13} + C_{22}C_{23} + C_{32}C_{33} = 0. \tag{11.4.10}$$

These six constraints are called the *redundancy relations* in the direction cosine formulation. It is noted that the relation in equation (11.4.8) preserves the orthogonality of the direction cosine matrix. There are two major features of this direction cosine formulation. First, it contains no singularity and is frequently used by aircraft engineers. Second, the direction-cosine formulation is not efficient from a computational point of view as numerical integration is excessively time-consuming.

On the other hand, in 1775, Euler provided an interesting description of the orientation of the noninertial coordinate frame relative to the inertial coordinate frame that can be described in terms of a single rotation through an angle Θ, about a particular axis, $\mathbf{E}$, which is usually known as the *Euler axis* (or the *eigenaxis*). The aircraft engineers frequently use the Euler axis rotation and the three Euler angles to develop the *Euler axis formulation* which consists of four-component description of orientation. The total rotation angle Θ and the three components of a vector $\mathbf{E}$ directed along the Euler axis $\mathbf{E}_x$, $\mathbf{E}_y$ and $\mathbf{E}_z$. Clearly, four parameters describe the orientation of an aircraft having only three degrees of freedom, and so, the Euler axis formulation is redundant as there are four degrees of freedom. In order to eliminate the redundancy, the length of the Euler axis vector is assumed to be unity so that $E_x^2 + E_y^2 + E_z^2 = 1$. Finally, the components of an arbitrary vector, $\mathbf{v}$ in the body-fixed coordinates (x_b, y_b, z_b) are related to the components of the same vector in the Earth-fixed coordinates (x_f, y_f, z_f) through the famous *Euler's formula* given by

$$\begin{bmatrix} v_{x_b} \\ v_{y_b} \\ v_{z_b} \end{bmatrix} = B \begin{bmatrix} v_{x_f} \\ v_{y_f} \\ v_{z_f} \end{bmatrix}, \tag{11.4.11}$$

where B is a 3×3 matrix given by

$$B = \begin{bmatrix} E_{xx} + C_\Theta & E_{xy} + E_z S_\Theta & E_{xz} - E_y S_\Theta \\ E_{xy} - E_z S_\Theta & E_{yy} + C_\Theta & E_{yz} + E_x S_\Theta \\ E_{xz} + E_y S_\Theta & E_{yz} - E_x S_\Theta & E_{zz} + C_\Theta \end{bmatrix}, \tag{11.4.12}$$

where $E_{ij} = E_i E_j (1 - C_\Theta)$, $C_\Theta = \cos\Theta$ and $S_\Theta = \sin\Theta$.

The inverse of the transformation matrix B in (11.4.11) is obtained by simply rotating through the negative of the total rotation angle Θ so that

$$\begin{bmatrix} v_{x_f} \\ v_{y_f} \\ v_{z_f} \end{bmatrix} = C \begin{bmatrix} v_{x_b} \\ v_{y_b} \\ v_{z_b} \end{bmatrix}, \tag{11.4.13}$$

where C is the matrix obtained from B by changing the sign of each S_Θ.

Thus, the velocity vector $(\dot{x}_f, \dot{y}_f, \dot{z}_f)$ of the aircraft relative to the Earth can be expressed as the sum of the wind velocity (w_x, w_y, w_z) relative to the Earth and the velocity of the aircraft (u, v, w) relative to the surrounding air in the form

$$\begin{bmatrix} \dot{x}_f \\ \dot{y}_f \\ \dot{z}_f \end{bmatrix} = C \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \tag{11.4.14}$$

where C is the matrix stated in (11.4.13).

Finally, the relation between the rate of change of the Euler axis rotation parameters and the body-fixed angular velocity vector (p, q, r) is given by

$$\begin{bmatrix} \dot{\Theta} \\ \dot{E}_x \\ \dot{E}_y \\ \dot{E}_z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2E_x & 2E_y & 2E_z \\ E'_{xx} + C/S & E'_{xy} - E_z & E'_{xz} + E_y \\ E'_{xy} + E_z & E'_{yy} + C/S & E'_{yz} - E_x \\ E'_{xz} - E_y & E'_{yz} + E_x & E' + C/S \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \tag{11.4.15}$$

where $E'_{ij} = -E_i E_j C/S$, $S = \sin\left(\frac{\Theta}{2}\right)$ and $C = \cos\left(\frac{\Theta}{2}\right)$. Thus, equations (11.4.14) and (11.4.15) represent the *kinematic transformation* equations in terms of the Euler axis rotation parameters. It is worthy to note that equation (11.4.15) has a singularity when $\sin\Theta = 0$, that is, at $\Theta = 0$ or π. So, the integrations of these equations is indeterminate. Thus, this analysis reveals that the existence of the singularity in the Euler axis formulation is a problem in aircraft flight simulation because it arises when fuselage axis is level with the ground and the aircraft is headed either due north or due south. Thus, the aircraft orientation is much more likely to occur in normal flight operation than the vertical orientations which give rise to the gimbal lock singularity involved in the Euler angle formulation. This is the reason why the Euler axis formulation is never applied to flight simulation. However, this formulation can be transformed into another singularity-free formulation by a change of variables.

We next make a change of variables to transform the Euler axis formulation to the *Euler–Rodrigues quaternion formulation*. The four parameters in the Euler axis formulation are utilized to introduce four new different parameters by

$$(e_0, e_x, e_y, e_z) = \left(\cos\frac{\Theta}{2},\ E_x \sin\frac{\Theta}{2},\ E_y \sin\frac{\Theta}{2}, E_z \sin\frac{\Theta}{2}\right). \qquad (11.4.16)$$

These four parameters are usually known as the *Euler–Rodrigues symmetric parameters*, and they form the basis of a very widely used description of orientation and rigid body rotation. Whether or not Euler knew of the Euler–Rodrigues symmetric parameters is a topic of historical debate. However, in 1770, Euler introduced four symmetric parameters for orthogonal transformations without the use of half angles. Roberson (1968) mentioned that Euler introduced a rotation matrix in terms of the so-called Euler–Rodrigues parameters.

Since the four parameters defined by (11.4.16) uniquely determine an orientation having only three degrees of freedom, these parameters must be related in some manner. This relation can be obtained by squaring the four parameters and adding them together so that

$$e_0^2 + e_x^2 + e_y^2 + e_z^2 = \cos^2\left(\frac{\Theta}{2}\right) + \left(E_x^2 + E_y^2 + E_z^2\right)\sin^2\frac{\Theta}{2}. \qquad (11.4.17)$$

In view of the fact that the Euler axis vector $E = (E_x, E_y, E_z)$ is a unit vector, it turns out from (11.4.17) that

$$e_0^2 + e_x^2 + e_y^2 + e_z^2 = 1. \qquad (11.4.18)$$

The Euler's formula (11.4.11) with (11.4.12) can be expressed in terms of half of the rotation angle by using the trigonometric identities, $\sin\Theta = 2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}$, $\cos\Theta = \cos^2\frac{\Theta}{2} - \sin^2\frac{\Theta}{2}$ and $1 - \cos\Theta = 2\sin^2\frac{\Theta}{2}$. Introducing the notation $S = \sin(\Theta/2)$, $C = \cos(\Theta/2)$ and $E_{ij} = 2E_iE_jS^2$ in (11.4.11) and (11.4.12), it follows that

$$\begin{bmatrix} v_{x_b} \\ v_{y_b} \\ v_{z_b} \end{bmatrix} = \begin{bmatrix} E_{xx} + C^2 - S^2 & E_{xy} + 2E_zSC & E_{xz} - 2E_ySC \\ E_{xy} - 2E_zSC & E_{yy} + C^2 - S^2 & E_{yz} + 2E_xSC \\ E_{xz} + 2E_ySC & E_{yz} - 2E_xSC & E_{zz} + C^2 - S^2 \end{bmatrix} \begin{bmatrix} v_{x_f} \\ v_{y_f} \\ v_{z_f} \end{bmatrix}. \qquad (11.4.19)$$

We next use the fact that $E_{ij} = 2e_ie_j$, and $S^2 = \left(E_x^2 + E_y^2 + E_z^2\right) S^2 = e_x^2 + e_y^2 + e_z^2$, and then put (11.4.16) into (11.4.19) to obtain the transformation equation

$$\begin{bmatrix} v_{x_b} \\ v_{y_b} \\ v_{z_b} \end{bmatrix} = D \begin{bmatrix} v_{x_f} \\ v_{y_f} \\ v_{z_f} \end{bmatrix}, \tag{11.4.20}$$

where D is a 3×3 matrix given by

$$D = \begin{bmatrix} e_x^2 + e_0^2 - e_y^2 - e_z^2 & 2(e_xe_y + e_ze_0) & 2(e_xe_z - e_ye_0) \\ 2(e_xe_y - e_ze_0) & e_y^2 + e_0^2 - e_x^2 - e_z^2 & 2(e_ye_z + e_xe_0) \\ 2(e_xe_z + e_ye_0) & 2(e_ye_z - e_xe_0) & e_z^2 + e_0^2 - e_x^2 - e_y^2 \end{bmatrix}. \tag{11.4.21}$$

It follows from (11.4.20) with (11.4.21) that the velocity of the aircraft is related to the ground speed and wind velocity by

$$\begin{bmatrix} \dot{x}_f \\ \dot{y}_f \\ \dot{z}_f \end{bmatrix} = D \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}. \tag{11.4.22}$$

Differentiating (11.4.16) with respect to time t gives the time rate change of the Euler axis rotation parameters in terms of the time rate of change of the Euler–Rodrigues symmetric parameters

$$\begin{bmatrix} \dot{e}_0 \\ \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} -S \\ E_xC \\ E_yC \\ E_zC \end{bmatrix} \frac{\dot{\Theta}}{2} + \begin{bmatrix} 0 \\ \dot{E}_xS \\ \dot{E}_yS \\ \dot{E}_zS \end{bmatrix}. \tag{11.4.23}$$

We next use (11.4.15) to express the time rate of change of the Euler axis rotation parameters in terms of the noninertial angular velocity vector so that (11.4.23) can be written as

$$\begin{bmatrix} \dot{e}_0 \\ \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \begin{bmatrix} -E_xS & -E_yS & -E_zS \\ C & -E_zS & E_yS \\ E_zS & C & -E_xS \\ -E_yS & E_xS & C \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{11.4.24}$$

which is, by (11.4.16),

$$\begin{bmatrix} \dot{e}_0 \\ \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -e_x & -e_y & -e_z \\ e_0 & -e_z & e_y \\ e_z & e_0 & -e_x \\ -e_y & e_x & e_0 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}. \tag{11.4.25}$$

Since equation (11.4.25) is linear in both the noninertial angular velocity vector and the Euler–Rodrigues symmetric parameters, it can be expressed in the form

$$\begin{bmatrix} \dot{e}_0 \\ \dot{e}_x \\ \dot{e}_y \\ \dot{e}_z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{bmatrix}\begin{bmatrix} e_0 \\ e_x \\ e_y \\ e_z \end{bmatrix}. \tag{11.4.26}$$

All these results reveal that equation (11.4.22), and equation (11.4.25) or equation (11.4.26) represent the kinematic transformation equations in terms of the Euler–Rodrigues symmetric parameters. It is worth noting that the Euler–Rodrigues formulation contains no singularities. The physical interpretation of the quaternion formulation is much less intuitive than that associated with the Euler angle. Moreover, there are several computational advantages over the Euler angle treatment or the direction cosine formulation. The computational advantage of the quaternion formulation can be increased even further through the use of Hamilton's quaternion algebra.

From a computational point of view, the Euler–Rodrigues quaternion formulation is far superior to either the Euler angle formulation or the direction cosine formulation. Numerical integration of the nine-component direction cosine formulation requires more than double the computation time needed for the four-component quaternion formulation. On the other hand, the Euler angle transformation requires about eleven times as long to evaluate as the quaternion transformation.

Chapter 12

Euler's Work on the Probability Theory

> "Euler devoted a portion of his universal interest to the study of the theory of risk and ... to questions involving the calculus of probability."

Louis Gustava du Pasquier

> "His industry and genius have left a permanent impression in every field of mathematics; and although his contributions to the Theory of Probability relate to subjects of comparatively small importance, yet they will be found not unworthy of his own great power and fame."

Isaac Todhunter

12.1 Introduction

During the sixteenth and seventeenth centuries, a great deal of attention was given to games of chances and gambling in general. An Italian nobleman suggested a problem of dice to Galileo, the solution of which is the first recorded result in the history of mathematical probability. A decade after Galileo's death, Pierre de Fermat and Blaise Pascal began their correspondence dealing with problems of throwing of a dice, arrangements of objects and chance of winning a game. They were both interested in the mathematical analysis of these problems including problems of gambling. Among other things, Pascal discovered the familiar formula for the binomial coefficient nC_r or $\binom{n}{r}$ and applied these results to solve the problem of points in the case where one player requires m points and the other n points to win a game. Interestingly, it was Christian Huygens who published the first treatise on the chances of winning games and problems of dice in 1657. This remained the best account of probability theory until the publication of Johann Bernoulli's first significant book '*Ars Conjectandi*' (*Art of Prediction*) on probability. In this treatise, Bernoulli presented the problems and solutions by Huygens and gave his own solution of them. He also introduced what are now known as *Bernoulli's trials*, and *the law of large numbers*. In 1711, De Moivre published another ground breaking book on probability, *The Doctrine of Chances*. In order to carry out the computations involved in the law of large numbers, it was De Moivre who approximated the binomial distribution with what is now known as the *normal* (or *Gaussian*) *distribution*. Some theoretical analysis of the applications of probability to hypothesis testing is due to British clergyman, Thomas Bayes (1702-1761) who stated the theorem that if an event has happened p times and failed q times, the probability that the chance of success will lie between the values a and b (all values are equally likely) is

$$\frac{1}{B(p,q)}\int_a^b x^p(1-x)^q dx, \tag{12.1.1}$$

where $B(p,q)$ is the Euler beta function. Bayes evaluated the above integral and the integral for the beta function in $(0,1)$ by approximation.

In an effort to discover new mathematical methods to solve major problems in probability theory, there emerged a great deal of work on permutations and combinations, summation of infinite series, finite differences and many new formulas for special functions. At the same time, a new major subject, the so called the *theory of errors*, arose associated with a set of observations (or measurements) of a trial of experiments due to the influence of astronomers, mathematical physicists and experimentalists. It was Laplace who fully recognized the major importance of probabilistic methods for investigating the results of measurements or observations. In his *Théorie analytique des probabilités* (*Analytic Theory of Probability*) published in 1812, Laplace gave the first systematic presentations of probability theory. He also proved that distribution of the average random observational errors which are uniformly distributed in an interval symmetric about the origin tends to the normal distributions as the number of observations increases to infinity. This celebrated result is now known as the *central limit theorem*. Based on the problems of games and the problem of experimental errors, Laplace established connection between these and the corresponding questions in mortality and life tables which provided the fundamental basis of insurance statistics. He also formulated the *error function*, $\text{erf}(x)$, defined by the integral

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt, \tag{12.1.2}$$

where the integral in (12.1.2) is called the *probability integral*, and $\operatorname{erf}(0) = 0$ and $\operatorname{erf}(\infty) = 1$.

The complementary error function, $\operatorname{erfc}(x)$ is also of special interest and defined by

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt, \tag{12.1.3}$$

so that $\operatorname{erfc}(0) = 1$ and $\operatorname{erfc}(\infty) = 0$.

Because of its importance in probability theory, the error function has been studied in detail and tabulated. So, it is of special interest in finding the asymptotic value of $\operatorname{erf}(x)$ for small or large x. When x is small, we can calculate the approximate value of $\operatorname{erf}(x)$ as follows:

We have

$$\int_0^x e^{-t^2}\, dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots\right) dt, \tag{12.1.4}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots . \tag{12.1.5}$$

Since this series is an alternating one, the sum of two successive terms gives an upper and lower limit to its sum. So, neglecting terms beyond x^7, the result would be deficient by an amount less than $\frac{x^9}{9.4!}$. If this is less than one in the fourth decimal places, then $\frac{x^9}{9.4!} < 10^{-4}$ or $x < 2 \times 10^{-2}$ approximately.

For large value of x, integrating by parts, we obtain

$$\int_x^\infty e^{-t^2}\, dt = \int_x^\infty \frac{1}{t} \cdot t\, e^{-t^2}\, dt = \frac{1}{2x} e^{-x^2} - \frac{1}{2} \int_x^\infty t^{-2} e^{-t^2}\, dt \tag{12.1.6}$$

$$= \frac{1}{2x}\, e^{-x^2} - \frac{1}{2^2.x^3}\, e^{-x^2} + \frac{1.3}{2.2} \int_x^\infty t^{-4} e^{-t^2}\, dt. \tag{12.1.7}$$

Continuing this process leads to the result

$$\int_x^\infty e^{-t^2}\, dt = \frac{e^{-x^2}}{2x} \left[1 - \frac{1}{2x^2} + \frac{1.3}{(2x^2)^2} - \frac{1.3.5}{(2x^2)^3} + \cdots\right]. \tag{12.1.8}$$

Since $\exp(-x^2)$ is a decreasing function in (x, ∞), the error involved in stopping at the fourth term is less than $\exp(-x^2) \int_x^\infty \frac{1.3.5.7}{2^4 t^8} dt$, that is, less than $\exp(-x^2) \frac{1.3.5}{2^4 x^7}$, the last term is retained. A similar result is given at any stage of the asymptotic expansion.

Based on a set of very general assumptions, Laplace established the *Method of Least Squares*. In fact, if the mean value of a set of observations is the most probable value, and positive errors are as likely as negative ones, the error function for the observations is of the Gaussian form

$$\frac{h}{\sqrt{\pi}}\, e^{-h^2x^2}, \tag{12.1.9}$$

where h is known as the *measure of precision* or called the *precision constant*. The value of h to be chosen is that which makes the probability maximum. If $x_1, x_2, \cdots, x_n$ are the measured observations of a number x, then the deviations of the observations from x are respectively $x - x_1$, $x - x_2, \cdots, x - x_n$. Assuming that the probability $p(x - x_k)$, $k = 1, 2, 3, \cdots, n$ is Gaussian, that is,

$$p(x - x_k) = \frac{h}{\sqrt{\pi}} \exp\left[-h^2(x - x_k)^2\right]. \tag{12.1.10}$$

Thus, the probability of all deviations is the product

$$\prod_{k=1}^{n} p(x - x_k) = \frac{h^n}{\pi^{n/2}} \exp\left[-h^2 \sum_{k=1}^{n} (x - x_k)^2\right]. \tag{12.1.11}$$

The problem is to find x so that probability is maximum which is equivalent to determining x so that $\sum_{k=1}^{n}(x-x_k)^2$ is a minimum. This gives the mean value $\bar{x}$ of $x_1, x_2, \cdots, x_n$. This method of finding the best value of an observation by assuming that the sum of the squares of the deviations from it will be a minimum is called the *Method of Least Squares*. Putting the mean value $\bar{x} = \frac{1}{n}\sum_1^{\infty} x_n$ in the right hand side of (12.1.11), and making the above probability maximum, it turns out that h is determined by the equation

$$\frac{d}{dh}\left[h^n \exp\left\{-h^2 \sum_{k=1}^{n} (\bar{x} - x_k)^2\right\}\right] = 0, \tag{12.1.12}$$

so that $\frac{n}{h} = 2h\sum_{k=1}^{n}(x_k - \bar{x})^2$. Therefore,

$$h^2 = \frac{n}{2\sum_{k=1}^{n}(x - x_k)^2} = \frac{1}{2\sigma'^2}, \tag{12.1.13}$$

where σ' is the standard deviation for the given set of observations defined by $n\sigma'^2 = \sum_{k=1}^{n}(x_k - \bar{x})^2$. It follows that the choice of h is such as to make the standard deviation for the observed set $x_1, x_2, \cdots, x_n$ coincide with that of the given population.

Thus, the works of Fermat, Pascal, Huygens, Laplace and Bernoulli marked the beginning of the theory of probability in the seventeenth and eighteenth centuries.

12.2 Euler's Work on Probability

Euler's modest amount of work on probability in finite samples spaces originated from the Genoese Lottery and the fundamental principle of counting. The word of lottery meaning the drawing of prizes 'by lots' was an ancient idea. The large scale lottery system initially originated in a small state of Genoa in Italy in 1643. This state became a part of the newly independent Kingdom of Italy after downfall of Napoleon Bonaparte (1769-1821) in 1815. However, the earlier roots of lottery system in Italy came from the City of Venice in 1520, and then from the city of Florence in 1530 where the first lottery system was introduced in the popular form of gambling in which lottery tickets were sold and cash prizes were awarded to winners. Subsequently, this became very popular in many Italian cities and then throughout Europe by the 1700s as a source of raising money for government. It was Benedetto Gentile of Genoa who was fully responsible for providing the major leadership role of the *Genoese Lottery System* for raising funds for the state. Even today, the Italian government uses a state lottery called "lotto" which involves drawing of five balls from a *ruota* (or *wheel*) containing balls numbered 1, 2, 3, $\cdots$, 90. In addition to European countries, the lottery system became popular in the United States of America by the early 1800s.

In 1749, an Italian business man named Roccolini approached Frederick the Great, then King of Prussia with a proposal to establish a lottery system involving the drawing of five numbers from 1 to 90. The King sent the proposal to his scientific advisor, Euler for a mathematical review concerning the implementation of a state lottery in Germany. At the royal request, Euler became very interested in analyzing the various aspects of the Genoese Lottery system and came up with an improved lottery system after addressing combinatorial issues in the analysis of this game of chance. Subsequently, the Berlin lottery was established in Germany in 1763.

In addition to his study and research on the royal assignment concerning mathematical problems of lotteries, Euler prepared seven notebooks during his early years in Basel, his first St. Petersburg period and his Berlin period. These notebooks contained solutions of many mathematical problems concerning probability theory and statistics, combinatorics, mortality, the mathematical theory of games, commercial as well as life insurance. He also investigated problems of pension, security, investment and interest. Under his direct supervision, his research assistant, N. I. Fuss published a textbook in 1776 covering basic elements of probability and statistics, insurance and organization of lotteries with tables. All of his work clearly provided a clear evidence of his life long interest in social sciences, probability theory and mathematical statistics.

Almost simultaneously, Euler wrote four mathematical papers on the probability of calculus based on solutions of various difficult questions in the Genoese Lottery. In addition, he made a presentation on Reflections on a singular type of lottery called the Genoese Lottery in 1763, and then published in 1862 in Euler's *Opera Posthuma I.* In this work, Euler introduced the hypergeometric distribution of the probability $p_{k,m}$ with parameters ℓ, n and k as

$$p_{k,m} = \frac{\begin{pmatrix} n \\ m \end{pmatrix}\begin{pmatrix} \ell - n \\ k - m \end{pmatrix}}{\begin{pmatrix} \ell \\ k \end{pmatrix}}, \tag{12.2.1}$$

where a player bets on k numbers to match m of them , ℓ and n are the parameters representing n tokens at random from a set numbered 1, 2, 3, $\cdots$, ℓ. Euler also gave a complete derivation of the desired probabilities for four problems with $k = 1, 2, 3, 4$ and $m = 0, 1, 2, 3$, and 4.

In his article "On the probability of sequences in the Genoese Lottery" that was presented to the Berlin Academy of Sciences in 1765, Euler investigated the sequences (or runs) of consecutive numbers that will appear among number drawn in a Genoese type lottery. Denoting a sequence of m consecutive numbers by (m), and introducing the idea of a *species*, he considered the solution of a general problem. Given an arbitrary parameter n, we denote the species s of a draw by

$$\alpha_1(\ell_1) + \alpha_2(\ell_2) + \cdots + \alpha_r(\ell_r), \tag{12.2.2}$$

so that

$$\sum_{i=1}^{r} \alpha_i \ell_i = n. \tag{12.2.3}$$

In addition, we denote $k = \sum_{i=1}^{r} \alpha_i$. Then the number of drawings that result in a species s is given by

$$\frac{(\ell - n + 1)(\ell - n)(\ell - n - 1)\cdots(\ell - n - k + 2)}{(\alpha_1!\,\alpha_2!\,\cdots\,\alpha_r!)}. \tag{12.2.4}$$

Thus, all desired probabilities can then obtained by dividing (12.2.4) by the total number of drawings $\begin{pmatrix} \ell \\ n \end{pmatrix}$.

In order to make a fair lottery system, Euler introduced m positive numbers so that $\sum_{r=1}^{m} \alpha_{m,r} = 1$ and distributed the lottery prize by the rule

$$f_{k,m} = \frac{\alpha_{k,m}}{p_{k,m}}. \tag{12.2.5}$$

Then the expected payoff for a lottery ticket costing one unit is

$$\sum_{m=1}^{k} p_{k,m}\, f_{k,m} = \sum_{k=1}^{n} \alpha_{k,m}. \tag{12.2.6}$$

However, the weights, $\alpha_{k,m}$ are not uniquely determined except $k = 1$. Therefore, for $k > 1$, Euler considered three possible weight distributions:
1. Uniform distribution,

$$\alpha_{k,m} = \frac{1}{k}, \tag{12.2.7}$$

2. Binomial distribution,

$$\alpha_{k,m} = \frac{1}{2^k - 1} \binom{k}{m}, \tag{12.2.8}$$

3. Modified Binomial distribution,

$$\alpha_{k,m} = \frac{1}{M_k}(k - m + 1) \binom{k}{m}, \tag{12.2.9}$$

where

$$M_k = \sum_{m=1}^{k} (k - m + 1) \binom{k}{m}. \tag{12.2.10}$$

In distributing weights, Euler was interested in minimizing the impact of large amount of prizes corresponding to large values of k.

In order to solve the problem for a given parameter n, we need a complete list of the various species, or *partitions* associated with a natural number n. The partition function $p(n)$ was discovered and studied by Euler in great detail in response to a 1740 letter from Phillip Naude. Thus, the probability question of enumerating and classifying the different species in the problem of runs in a lottery drawing reduces to the problem of partitions of numbers.

Although combinatorics became a new modern branch of mathematics fairly recently, problems of counting have a long and early history. Euler considered problems of permulations and combinations and formulated the problem as follows. Given any series of n letters a, b, c, d, e, $\cdots$, to find how many ways they can be rearranged so that none returns to the position it initially occupied. In this context, Euler introduced the notation $\Pi(n)$ to represent the number of permulations of the n letters a, b, c, d, e, $\cdots$, in which none occupies its original position. Such a *permulation* is now known as a *derangement*. Using some simple and ingeneous argument, he also proved several recursion formulas for $\Pi(n)$ including the double recursive formula

$$\Pi(n) = (n-1)\left[\Pi(n-1) + \Pi(n-2)\right]. \tag{12.2.11}$$

This reduces to the simple formula

$$\Pi(n) = n\,\Pi(n-1) + (-1)^n \quad \text{for} \quad n \geq 2. \tag{12.2.12}$$

He also derived a remarkable closed-form expression for $\Pi(n)$, $n > 1$, as

$$\Pi(n) = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + \frac{(-1)^n}{n!}\right]. \tag{12.2.13}$$

Euler considered the question: if an ordered set of objects is randomly permuted, what is the *probability* that none of the objects returns to the original position? In other words, the major problem is to find the probability p_n of derangements $\Pi(n)$ so that none of the objects being returned to its original position. Thus, the probability p_n is given by

$$p_n = \frac{\Pi(n)}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}. \tag{12.2.14}$$

It is important to find p_n for small and large values of n. In 1751, Euler calculated the limit of p_n as $n \to \infty$ as

$$\begin{aligned}\lim_{n\to\infty} p_n &= \lim_{n\to\infty}\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right] \\ &= \frac{1}{e} \approx 0.36787944\ldots,\end{aligned} \tag{12.2.15}$$

where e is the universal exponential constant. In other words, the likelihood of a derangement converges very rapidly to e^{-1}. So, the involvement of the constant e in the combinatorial problem is another remarkable fact.

In 1779, Euler presented a paper on "A curious question from the doctrine of combinations". This paper had also some impact on the probability of winning a game, that is, on the lottery problems, where the number of combinations, nC_r of n objects selected r at at time with the order of the chosen objects is *not* taken into account. On the other hand, the number of permutations, nP_r of n objects taken r at a time with the order of chosen objects is taken into account so that $nP_r = r!\; nC_r$.

12.3 Euler's Beta and Gamma Density Distributions

In probability and statistics, the Euler beta function is used to define the beta density distribution $B_{m,n}(x)$ in $0 < x < 1$:

$$B_{m,n}(x) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\,(1-x)^{m-1}x^{n-1}, \tag{12.3.1}$$

where $m > 0$ and $n > 0$ are parameters.

Obviously,

$$\int_0^1 B_{m,n}(x)\,dx = 1. \tag{12.3.2}$$

The expectation μ and the variance σ^2 of the beta distribution are given by

$$\mu = \int_0^1 x\,B_{m,n}(x)dx = \frac{n}{m+n}. \tag{12.3.3}$$

$$\begin{aligned}\sigma^2 &= \int_0^1 (x-\mu)^2 B_{m,n}(x)dx = \int_0^1 x^2 B_{m,n}(x)dx - \mu^2 \\ &= \frac{mn}{(m+n)^2(m+n+1)}.\end{aligned} \tag{12.3.4}$$

For $m > 1$ and $n > 1$, the graph of $B_{m,n}(x)$ is bell-shaped. If $m < 1$ and $n < 1$, the graphs of $B_{m,n}(x)$ is U-shaped, tending to infinity at the limits.

A simple modification of the beta density distribution is defined by

$$\frac{1}{(1+x)^2}B_{m,n}\left(\frac{1}{1+x}\right) = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\,\frac{x^{m-1}}{(1+x)^{m+n}},\ 0 < x < \infty. \tag{12.3.5}$$

In particular, when $m = \frac{1}{2}$ and $n = \frac{1}{2}$, the $B_{\frac{1}{2},\frac{1}{2}}(x)$ occurs frequently in fluctuation theory

$$B_{\frac{1}{2},\frac{1}{2}}(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \qquad 0 < x < 1. \tag{12.3.6}$$

This is so called the *arc sine density distribution* in $0 < x < 1$.

In his 1968 study of string theory, G. Veneziano first recognized that data for scattering of particles could be fit well with an amplitude which is the sum of beta functions. This representation naturally occurred in a depiction of particles as strings rather than points.

On the other hand, the Euler gamma function is used to represent the probability density distribution function in $0 < x < \infty$ in the form

$$f_{\alpha,\nu}(x) = \frac{1}{\Gamma(\nu)}\,\alpha^\nu x^{\nu-1}e^{-\alpha x}, \tag{12.3.7}$$

where $\alpha > 0$ is the trivial parameter and $\nu > 0$ is an essential parameter. The particular case $f_{\alpha,1}(x) = \alpha\, e^{-\alpha x}$ *exponential* density distribution with the expectation α^{-1} and the variance α^{-2}, and $f_{\alpha,n}(x)$ is given by

$$f_{\alpha,n}(x) = \frac{\alpha}{n!}\,(\alpha x)^n\, e^{-\alpha x}. \tag{12.3.8}$$

A simple calculation shows that the expectation and the variance of the gamma distribution are (ν/α) and (ν/α^2) respectively. The family of gamma density distributions is closed under convolution operation

$$f_{\alpha,\mu}(x) * f_{\alpha,\nu}(x) = f_{\alpha,\mu+\nu}(x) \quad \mu > 0,\ \nu > 0, \tag{12.3.9}$$

where the convolution is defined in $(0,\infty)$ by

$$f_{\alpha,\mu}(x) * f_{\alpha,\nu}(x) = \frac{\alpha^{\mu+\nu}}{\Gamma(\mu)\Gamma(\nu)}\, e^{-\alpha x} \int_0^x (x-y)^{\mu-1} y^{\nu-1} dy. \tag{12.3.10}$$

After the change of variable $y = xt$, this expression differs from $f_{\alpha,\mu+\nu}(x)$ by a numerical constant only and this is equal to unity because both $f_{\alpha,\mu+\nu}(x)$ and (12.3.10) are probability density distribution functions.

It follows from (12.3.7) that the graph of $f_{1,\nu}(x)$ is clearly monotonic if $\nu \le 1$, and unbounded in the neighborhood of the origin when $\nu < 1$. For $\nu > 1$, the graph of $f_{1,\nu}(x)$ is bell shaped with the maximum value $(\nu-1)^{\nu-1}\exp\left[-(\nu-1)\right]/\Gamma(\nu)$ attained at $x = \nu - 1$. Using the Stirling approximation (8.2.58), the maximum value tends to $\sqrt{2\pi(\nu-1)}$ for large $(\nu-1)$. It follows from the central limit theorem that

$$\sqrt{\frac{\nu}{\alpha}}\, f_{\alpha,\nu}\left(x\sqrt{\nu/\alpha}\right) \to \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x^2\right) \quad \text{as} \quad \nu \to \infty. \tag{12.3.11}$$

In other words, the gamma density distribution tends to the normal density distribution as $\nu \to \infty$.

In general, if $f(x)$ is a continuous probability density function, then the random variable X has distribution f if for any interval $[a,b]$ the probability that X assumes a value in $[a,b]$ is

$$P\left(X = x, a \le x \le b\right) = \int_a^b f(x)dx. \tag{12.3.12}$$

If $X_1,\ X_2,\ \cdots,\ X_n$ are n independent random variables with the same mean μ and the same variance σ^2 (or the same standard deviation σ), then their sum $S_n = (X_1 + X_2 + \cdots + X_n)$ has the mean $n\mu$ and variance $n\sigma^2$. The *standardized random variable* Z_n with mean zero and variance one is defined by

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{1}{\sigma\sqrt{n}}\sum_{n=1}^{n}(X_r - \mu). \tag{12.3.13}$$

The fundamental *Central Limit Theorem* in probability and statistics states that, as $n \to \infty$, Z_n tends to a standard normal distribution. In other words, the distribution function $F_n(z)$ of Z_n satisfies

$$\lim_{n\to\infty} F_n(z) = \lim_{n\to\infty} P(Z_n \le z) = F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} dx. \quad (12.3.14)$$

Another version of the *Central Limit Theorem* is as follows. If $F(z)$ is a distribution function with zero mean and unit variance, then

$$\lim_{n\to\infty} F^{n*}(z\sqrt{n}) = F(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2} dx, \quad (12.3.15)$$

where $F^{n*} = F * F * \cdots * F$ is the convolution of F with itself n times and $(F * G)(z)$ is the *convolution* of F and G defined by

$$(F * G)(z) = \int_{-\infty}^{\infty} F(z-y)G(y)dy, \quad (12.3.16)$$

provided the integral exists.

Finally, we use the method of probability to derive the celebrated Wallis product formula from the student t-distribution with ν as the number of degrees of freedom given by

$$f_\nu(x) = a_\nu \left(1 + \frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} \quad (12.3.17)$$

where student was the pen name of William Gosset (1876-1937) and

$$a_\nu = \Gamma\left(\frac{\nu+1}{2}\right) / \sqrt{\pi\nu}\, \Gamma\left(\frac{\nu}{2}\right), \quad (12.3.18)$$

$\nu(> 0)$ and x are any real numbers. Evidently, the student distribution is a continuous probability distribution in $(-\infty, \infty)$. In other words,

$$a_\nu \int_{-\infty}^{\infty} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} dx = 1. \quad (12.3.19)$$

This follows from the substitution of $x = \sqrt{\nu}\tan\theta$ or $dx = \sqrt{\nu}\sec^2\theta d\theta$ in (12.3.19) so that

$$a_\nu \int_{-\infty}^{\infty} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} dx = a_\nu\, 2\sqrt{\nu} \int_0^{\pi/2} \cos^{\nu-1}\theta\, d\theta$$

$$= a_\nu \sqrt{\nu} B\left(\frac{\nu}{2}, \frac{1}{2}\right) = a_\nu \sqrt{\nu}\, \frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)} = a_\nu \frac{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)} = 1. \quad (12.3.20)$$

Using the fact that

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n, \quad (12.3.21)$$

it follow that

$$\lim_{\nu\to\infty}\left(1+\frac{x^2}{\nu}\right)^{-\nu/2} = \left(e^{x^2}\right)^{-\frac{1}{2}} = e^{-x^2/2}. \tag{12.3.22}$$

Thus, it follows from (12.3.20) that

$$\begin{aligned}\lim_{\nu\to\infty}\frac{1}{a_\nu} &= \lim_{\nu\to\infty}\int_{-\infty}^{\infty}\left(1+\frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)}dx \\ &= \int_{-\infty}^{\infty}\lim_{\nu\to\infty}\left(1+\frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)}dx = \int_{-\infty}^{\infty}e^{-x^2/2}dx = \sqrt{2\pi}.\end{aligned} \tag{12.3.23}$$

More explicitly,

$$\lim_{\nu\to\infty} a_\nu\left(1+\frac{x^2}{\nu}\right)^{-\frac{1}{2}(\nu+1)} = \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}. \tag{12.3.24}$$

This means that the student t-distribution tends to a standard normal as $\nu \to \infty$.

Letting $\nu = 2n$ and using $\Gamma(n+1) = n\,\Gamma(n) = n!$, we obtain without limit

$$\begin{aligned} a_{2n} &= \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\sqrt{2\pi n}\,\Gamma(n)} = \frac{(2n-1)(2n-3)\cdots 5.3.1}{\sqrt{2\pi}.(n-1)!}\cdot\frac{\sqrt{\pi}}{2^n} \\ &= \frac{1.3.5.\cdots(2n-3)(2n-1)}{2.4.6.\cdots(2n-2).2n}\cdot\sqrt{\frac{n}{2}} \\ &= \left[\frac{1.3}{2.2}\cdot\frac{3.5}{4.4}\cdots\frac{(2n-1)(2n+1)}{2n.2n}\cdot\frac{1}{(2n+1)}\right]^{\frac{1}{2}}\cdot\sqrt{\frac{n}{2}} \\ &= \left[\prod_{n=1}^{n}\frac{(2n-1)(2n+1)}{2n.2n}\right]^{\frac{1}{2}}\cdot\left(\frac{n}{4n+2}\right)^{\frac{1}{2}}. \end{aligned} \tag{12.3.25}$$

We next take the square of both sides of (12.3.25) and then perform the limit of its reciprocal expression as $n \to \infty$ to obtain the Wallis product formula

$$\prod_{n=1}^{\infty}\frac{2n.2n}{(2n-1)(2n+1)} = \lim_{n\to\infty}\left(\frac{n}{4n+2}\right)\left(\frac{1}{a_{2n}^2}\right) = \frac{\pi}{2}. \tag{12.3.26}$$

Chapter 13

Euler's Contributions to Ballistics

> "Although to penetrate into the intimate mysteries of nature and thence to learn the true causes of phenomena is not allowed to us, nevertheless it can happen that a certain fictive hypothesis may suffice for explaining many phenomena."

Leonhard Euler

> "It is today quite impossible to swallow a single line of d'Alembert, while most writings of Euler can still be read with delight."

Carl Gustav Jacobi

13.1 Introduction

The ballistics revolution is usually attributed to two men, Leonhard Euler and Benjamin Robins, a British military engineer, who based their works on the early contributions of Galileo to the motion of a projectile in the atmosphere without taking into effects of air resistance. In early days of ballistics, the parabolic trajectory of a projectile was developed for the study of the motion of cricket balls or cannon balls. Like Euler, Robins was born in 1707. As early as 1727, Euler spent a considerable amount of time to study mechanics. Indeed, he published his remarkable two-volume treatise on *Mechanics.* The two final chapters of his first volume contained a mathematical treatment of circular motion of a particle in a vacuum and in a resisting medium. Euler derived the differential equations of motion of a particle along the tangent and normal to the trajectory at any point. Based on the motion of a particle under the action of a central force formulated in Newton's universal law of gravitation, Euler made an extensive investigation of the problem to the extent that it is appropriate to recognize him as one of the founders of analytical celestial mechanics. Almost simultaneously, Euler solved another fundamental problem which dealt with the curvilinear motion of a particle in a resisting medium under the action of gravity, that is, the major problem of exterior ballistics. In fact, in 1753, he published his indepth study in a 40-page article entitled, "Investigation of the actual curve described by a body projected into the air or some other medium." Euler reported that there exists some experimental and theoretical evidence that the air resistance is proportional to the square of the body's velocity in the atmosphere. In order to develop a fairly general mathematical theory of projectile, he recognized that three forces always act on a projectile including the vertically downward acceleration of gravity, the upward directed buoyancy force, and the resistance of the fluid against the direction of motion. However, the buoyancy force is usually small in air, but significantly large in a medium like water. This work clearly demonstrated his early research and interest in ballistics.

On the other hand, in 1742, Robins published his original discoveries in his book on *New Principles of Gunnery* containing the determination of the force of gun-powder and an investigation of the difference in the resisting power of the air to swift and slow motions that represented his great achievement to develop the modern research in ballistics. In 1783, John Pringle (1707-1782), the President of the Royal Society of London, put it more simply by stating that Robins created a "new science". John Nef (1899-1988), an author of a book entitled *War and Human Progress: An Essay on the Rise of Industrial Civilization*, wrote in 1950 that Robins' work "provides a landmark in the interrelations between knowledge and war".

On the other hand, Thomas P. Hughes (1822-1896) said that Robins was "a founder of modern gunnery", and Charles Hutton (1737-1823), a British engineering professor, acclaimed that Robins' research represents "the first work that can be considered as attempting to establish a practical system of gunnery, and projectiles, on good experiments, on the force of gun powder, on the resistance of air, and on the effects of different pieces of artillery".

Among many pioneering contributions of Robins to ballistics, his single most achievement was his discovery of the ballistics pendulum, the first reliable and revolutionary scientific instrument was invented for measuring speed of a musketball in particular and a projectile in general. He used this instrument to discover enormously complicated air-resistance forces acting on projectiles moving with a very high speed. It is interesting to point out that Robins gave a demonstration of these air-resistance measurements before the Royal Society of London that showed a significant modification of the parabolic trajectory moving at a very high speed. He also demonstrated that air resistance at subsonic and supersonic velocities follow different rules and pointed out the importance of a sonic barrier and was well aware of the changes in rules for calculating air resistance at velocities above or below it. According to Huygens and Newton, the air resistance is proportional to the square of the velocity of a projectile, Robins showed that this was correct only at lower velocities. At velocities greater than the speed of sound (1087 feet per second) in air, he showed that the resistance is increased by a factor of three. Robins also discovered another aerodynamic property which deals with the lateral deflection of moving projectiles as this phenomenon is readily observed while playing tennis or baseball. He also observed the spin imparted on the musket ball as it struck the barrel side of the musket during firing as the cause of its deflection. Based on his experimental proof of his theory with a musket, Robins showed that the bullet reversed its lateral direction of motion and moved to the right side of the musket. He then elucidated experimentally this phenomenon by noting that the deflected musket forced the bullet to rotate from the left to the right due either to spin or to the difference between the geometrical center of mass of spherical bullet or its actual asymmetrical center of mass which is known as the *Robins Spin Effect* or the *Magnus Effect*, since G. Magnus (1802-1870) German physicist, also investigated it after a century later with full knowledge of Robins' experimental observations of the drift of the trajectory of a musket ball. Another work of Robins' dealt with the understanding of the Robins effect to explain theoretically the better accuracy of rifles over muskets. He made a presentation of his work entitled, "Of the Nature and Advantage of Rifled Barrel Pieces" at the Royal Society in 1747. It was Robins who first organized the tabulated numerical solutions of differential equations of motion of a projectile in the atmosphere due to Euler. Robins presented his ballistic table to the Royal Society in 1746 just a year after Euler's translation of Robins book on *New Principles of Gunnery* in 1745, and prepared an approximate ballistics tables, but it is sufficiently accurate for projectile motion as compared with experimental observations. All ballistics tables suggest the war power available to artillery and military officers after ballistics revolution. While investigating the behavior of bullets from muskets and rifles, Robins formulated the fundamental principles of aerodynamics. In summary, Robins fundamental research revolutionized experimental and theoretical ballistics by transforming it into an aerodynamic and thermodynamic science. So, his work represents basic research in science and engineering. It may not be out of place to mention that Benjamin Robins was appointed as Engineer General of India and Captain of the Madras artillery for the East India Company in December 8, 1749 and arrived in at Fort St. David on July 14, 1750. His duties and responsibilities included improvements of proper training in its use of cannons and mortars and proper allotment of ammunition. In addition to commanding the artillery batteries, he redesigned Fort St. David as a part of the East India Company's military buildup against the French after the War of the Austrian Succession in a short period of one year. Unfortunately, he died of a fever on July 29, 1751 in India. During those twelve months, he had a difficult time to get used to the new climate, new job and challenges, and new colleagues in India, while he had made a definite plan to survey the country of India and the coast accurately.

So, this short chapter is devoted to describe Euler's major contributions to ballistics. In a short period of eleven years, both Euler and Robins dramatically expanded the mathematical and scientific knowledge of ballistics by constructing mathematical and empirical foundations. So, their works represent a scientific revolution in ballistics.

13.2 Euler's Research on Ballistics

If x and z are horizontal and vertical coordinates of a position of a particle of mass m under the action of a constant gravitational force, $-mg$ and if (u, w) are the horizontal and vertical velocities of the particle with the initial values (u_0, w_0), then there is no acceleration in the x direction so that $u = u_0$ and $\frac{df}{dt} = u_0 \frac{df}{dx}$ for any differentiable function f. The vertical equation of motion of the particle is

$$\frac{d^2z}{dt^2} = u_0^2 \frac{d^2z}{dx^2} = -g. \tag{13.2.1}$$

Integrating this equation twice with the initial position $(x, z) = (0, 0)$ gives the celebrated parabolic trajectory

$$z = -\left(\frac{g}{2u_0^2}\right) x^2 + \left(\frac{w_0}{u_0}\right) x. \tag{13.2.2}$$

In his work on the projectile theory, Galileo admitted that his theory without the effect of air resistance was inaccurate for high speed projectiles. However, for heavy mortar shells moving with low velocities the air resistance is too small to decelerate the projectile significantly during its short flight in the words of Galileo:

"This excessive impetus of violent shots can cause some deformation in the path of a projectile, making the beginning of the parabola less tilted and curved that its end. But this will prejudice our Author little or nothing in practicable operations, his main result being the compilation of a table of what is called the "range" of shots, containing the distances at which balls fired at (extremely) different elevations will fall. Since such shots are made with mortars charged with but little powder, the impetus is not supernatural in these, and the (mortar) shots trace out their paths quite precisely."

Euler was familiar with this unrealistic model of the motion of a projectile due to Galileo and was aware of all assumptions made in Galileo's projectile theory which neglected the significant air resistance on the projectiles and the fact that the parabolic arc was an overestimation of the range of the trajectory. So, Galileo made an appropriate comments on the importance of air resistance in the basic problem of ballistics as follows:

"This formula for [the constant of proportionality of the resistance force] will hold when the movement of the ball is not too fast so that the air can quite freely fill the space which the ball has left behind. But if the movement is so rapid that the air is unable to occupy instantaneously the space which the ball had occupied, so that this space remains empty, at least for an instant, then the front part of the ball is subject to the atmospheric pressure which, not being balanced by an equal pressure behind, it is clear that the resistance will be increased by the entire atmospheric pressure on the anterior of the ball."

At the same time, Euler had already made some major contributions to mechanics and so, he was naturally interested in the area of ballistics. In response to a royal assignment by Frederick the Great, and stimulated, by Robins' famous work, Euler first translated Robins' 150-page book on *New Principles of Gunnery* into German in 1745 with a large and extensive mathematical commentaries that the translation was over five times (720 pages) as long as Robins' original book. Robins also developed experimental methods to measure ballistic quantities and determined the speeds and trajectories of musket balls with deep theoretical insights. Christian Huygens, Newton and Johann Bernoulli made a serious attempt to study projectile motion in a resisting medium. They were not successful to improve the theory of Galileo for solving gunnery problems because the basic differential equations of projectile motion in the atmosphere are *nonlinear* and they do not have an exact solution. Using Newton's basic assumptions that a force opposing the motion proportional to the square of the velocity is to be added to the gravitational force g, Johann Bernoulli formulated the equations of motion

$$\ddot{x} = -k\left(\dot{x}^2 + \dot{z}^2\right)^{\frac{1}{2}} \dot{x}, \tag{13.2.3}$$

$$\ddot{z} = -g - k\left(\dot{x}^2 + \dot{z}^2\right)^{\frac{1}{2}} \dot{z}, \tag{13.2.4}$$

where x and z are horizontal and vertical coordinates and k is a constant. Euler recognized that the coupled system of ordinary differential equations is difficult to solve analytically, he developed approximation methods to determine approximate solutions. In his translation of Robins' book, Euler added a large amount of mathematical commentaries in his translation of 720-page book. Some of his insightful comments led to new and modern approach to ballistics, especially, to the study of an air flow over the bullet and the Robins' spin effect. Even without spin, the motion of a rigid sphere of radius a with velocity U moving through a fluid of kinematic viscosity ν is extremely complicated and depends on the behavior of boundary layers around the sphere.

Even in the absence of spin effect, the problem of a sphere moving in a fluid is enormously complicated and depends on the nature of boundary layers around the sphere. The drag force, D is usually characterized by a *drag coefficient*, C_D which depends on the Reynolds number $Re = \frac{2aU}{\nu}$ of the flow, where for a sphere of radius $r = a$ moving with velocity U in a fluid (air) with viscocity ν. The drag coefficient of a projectile with projected area $A = \pi a^2$ in the direction of motion due to a fluid of density ρ is then given by

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 A} = \frac{D}{\frac{1}{2}\rho U^2 \pi a^2}. \tag{13.2.5}$$

On the other hand, the Stokes force acting on the sphere due to the idealized smooth flow of a viscous fluid around the sphere is given by

$$D = 6\pi\mu a U. \tag{13.2.6}$$

Substituting the value D from (13.2.6) into (13.2.5) and using the definition of the Reynolds number yields

$$C_D = \frac{24}{Re}. \tag{13.2.7}$$

As the Reynolds number increases the flow separates from the sphere and eddies are formed at the downstream end of the sphere. Separation initially occurs near the stagnation point at the back of the sphere, but moves out as the Reynolds number increases with drag coefficients C_D attaining an approximate value one. In flow around a sphere at about $Re \sim 5 \times 10^5$ or more, the boundary layer makes a transition to turbulence before separating from the surface. Separation is then delayed because of the enhanced mixing effect due to turbulence, until a point is attained where a much stronger retardation of the external flow has occurred. The wake is much narrower and produces significantly less changes in the external flow. There is a significant decrease in the drag coefficient, C_D as observed by Achenback (1972) in his major article on "Experiments on the flow past spheres at very high Reynolds numbers". In fact, many other experiments in the nineteenth and twentieth centuries confirmed the quantitative and qualitative nature of the drag coefficients. For more information on drag coefficients, the reader is referred to a recent paper of Miller and Baily (1979) who described a fascinating account of experimental works on the drag coefficients during the nineteenth and twentieth centuries.

For flow around spheres, the major changes between the two flow regimes just stated above occurs at the so-called the *critical Reynolds number* whose actual value (between 10^5 and 5×10^5) depends upon various factors that may tend to promote or to lessen disturbances in the associated boundary layers. Some properties of the flow characteristics and the dramatic decrease in the drag coefficient C_D from value around 0.5 for subcritical Reynolds numbers to values around 0.1 for supercritical Reynolds numbers have been extensively exploited by advanced players of ball games.

The velocity potential for irrotational flow around a sphere is often written in terms of a spherical polar coordinates θ so that $z = r\cos\theta$ and $s = r\sin\theta$. In a frame of reference in which the fluid far away from the sphere is at rest so that the sphere is moving through the fluid in the negative z direction, the velocity potential is obtained by subtracting the uniform-stream velocity potential Uz or $Ur\cos\theta$ so that the velocity potential is

$$\phi = U\left(1 + \frac{1}{2}a^3 r^{-3}\right) r\cos\theta - Ur\cos\theta \tag{13.2.8}$$

$$= \frac{1}{2}Ua^3 r^{-2}\cos\theta. \tag{13.2.9}$$

The velocity potential for the steady flow past a stationary sphere is given by the first term of the right hand side of (13.2.8) and it fails to represent accurately the steady flow. Similarly, the potential (13.2.9) fails to describe the fluid flow associated with steady movement of the sphere in the fluid.

But the potential represents the dipole field (13.2.9) due to a sphere set impulsively into motion in the fluid. These conclusions forshadow some of the more general results of Chapter 8 in Lighthill's modern book (1986).

Euler's extensive comments on Robins' work revealed new insights on supersonic flow, where the velocity of projectile is faster than the sound velocity at which pressure waves move in the fluid, and possible development of shock waves. In his commentaries, Euler praised Robins' book and also criticized some of Robins' assumptions and approximations, pointed out his mathematical errors and added many major areas of artillery and ballistics not covered in Robins' book. For example, Euler made a mathematical analysis of a ballistics trajectory which incorporated the effects of air resistance on projectiles. He also published an early study of a pressure vessel and the theoretical strength of a gun barrel. He formulated the different equations for a pressurized cylinger with unconstrained ends where the maximum stress is the product of the internal pressure and the ratio of the radius of the cylinder to its wall thickness. In his work, he gave a first proof of famous *d'Alembert's paradox* in fluid mechanics that an inviscid potential flow in three-dimensions around a rigid body moving at a uniform velocity exerts no resistive force on the body. Mathematically, $\mathbf{F} = -(d\mathbf{P}/dt) = 0$, where $\mathbf{P} = m\mathbf{v}$ is the total momentum of the fluid and $\mathbf{F}$ is the total external force transmitted to the fluid by the body. However, the behavior of the experimentally predicted flow is quite different from that of the potential flow. This fallacy lies not in the direct neglect of viscous forces, but rather in the assumption that there is *no* vorticity in the fluid outside the body. A body moving through a real fluid has behind it a wake containing vorticity. If the flow around a steadily moving body could be made quite close to a potential flow, the resistive force would become very small, but *not* zero.

In addition, Euler made a pioneering contributions to mathematical analysis of supersonic air resistance, Euler developed approximate methods to simplify his analysis of the complicated differential equations of ballistics motion. However, his solution was correct only for projectiles moving with low velocities. Based on the trapezoidal rule, he numerically integrated the equations of motion representing the trajectory's range, time, velocity, and altitude. He then prepared ballistics tables for a projectile fired at certain muzzle velocities and elevation angles. Subsequently, Euler's ballistics tables were expanded to analyze more complicated ballistics trajectories for high-speed and long-range artillery during the nineteenth and early twentieth century.

The new scientific ground opened by both Euler's and Robins' ballistics revolution was rapidly utilized by the European military organizations. Anne Robert Jacques Turgot (1727-1781) wrote to Louis XVI in 1774 that "the famous Leonhard Euler, one of the greatest mathematicians of Europe, has written two works which could be very useful to the schools of the navy and the artillery. One is a treatise on the construction and Maneuver of Vessels, the other is a commentary on the principles of artillery of Robins'.... I propose that your majesty order these to be printed." During the Napoleonic Wars, more Ecole Polytechnique graduates served in the artillery than in any other branch of the French military. Napoleon Bonaparte's artillery professor, Jean-Louis Lombard (1723-1794), translated Euler's and Robins' work into French in order to prepare cadets with sufficient knowledge and information about ballistics with special reference to major research work in gun and projectile performance. Napoleon was very familiar with Euler's and Robins' ballistics works, the Robin effect, the limitation of Galileo's parabolic projectile theory, and the effect of air-resistance on projectiles, and wrote his twelve-page summary of *New Principles of Gunnery* in 1788. Based on his thorough knowledge and competence in ballistics, Napoleon wrote two memoirs on this subject. Although it is difficult to measure the actual impact of Euler and Robins on Napoleon, their ideas and research significantly contributed to Napoleon's great success as the military commander in chief, especially in his utilization of artillery. Napoleon is probably most famous for his military achievements because of his tremendous expertise and sound knowledge of ballistics. Indeed, the ballistics revolution made a direct impact on the combat of the French Revolutionary War in the 1790's. The French military officers utilized ballistics tables in combat in both the French Revolutionary War and the Napoleonic Wars.

Almost simultaneously, Great Britain was also stimulated in part by the ballistics revolution to increase the scientific and mathematical education of artillery officers. In 1741, the British created a new military academy at Woolwich in Kent for the education of artillery and engineering officers with the major focus on the combination of theory and practice. Charles Hutton used his own textbook *A course of Mathematics, for the use of the Gentlemen Cadets in the Royal Military Academy at Woolwich* to teach sufficient mathematics to understand Euler's and Robins' research work in ballistics. He also adopted Robins' *New Principles of Gunnery* as a textbook when teaching ballistics during his thirty-four year tenure at the Royal Military Academy at Woolwich. Indeed, the entire Europe and the United

States of America were inspired by the ballistics revolution to increase the mathematical and scientific education and training of artillery and military officers.

Based on Galileo's celebrated work, both Euler and Robins made tremendous progress on research in both mechanics and ballistics. Both experimental and theoretical works on modern ballistics with ballistics tables of Euler and Robins provided the fundamental basis of ballistics research in science and technology. Like any pioneering effort, their research work was imperfect, yet it provided a fairly rigorous mathematical and scientific foundation for ballistics.

Chapter 14

Euler and his Work on Astronomy and Physics

> "... the celebrated three-body problem, which arises from the study of lunar motion, is still too far beyond the power of analysis for one to be able to hope to find a complete solution."

Leonhard Euler

> "From this I draw the undeniable conclusion that one cannot hope to solve the general case of the three-body problem while no means is known for solving it even in the case where the three-bodies move along one and the same line."

Leonhard Euler

> "All celebrated mathematicians now alive are his disciples: there is no one who ... is not guided and sustained by the genius of Euler."

Marquis de Condorcet

14.1 Introduction

Historically, astronomy is one of the oldest subject in natural sciences. In 1543, Nicolaus Copernicus of Poland published a book describing the Solar System with the idea that the Earth moved around the Sun without a proof. During the late 1500s, the Danish astronomer, Tycho Brahe successfully proved the theory of Copernicus and provided an accurate records of showing the positions of the stars and planets. This was followed by the famous discovery of German mathematical scientist, Johann Kepler who described the observed motion of the planets in elliptical orbit. He also first formulated the three laws of planetary motion. In order to provide description of the Solar System consisting of the Sun and all the heavenly bodies that revolve around it based on observations, Galileo Galilei made many sensational discoveries in astronomy and mechanics which added support to the Copernicus Solar System. In 1638, he published a famous book entitled *Dialogues Concerning Two New Sciences* which contained his whole life's work on motion, velocity, acceleration and gravity and first formulated a basis for the three laws of motion which discovered later by Newton in 1687. Indeed, Kepler's three laws of planetary motion also formed an indispensable basis of Newton's discovery of universal gravitation.

During the 1660s, Sir Isaac Newton, a British mathematical scientist and philosopher, discovered the fundamental mathematical and physical laws of nature which were included in his greatest book of *Philosophiae Naturalis Principia Mathematica* (*The Mathematical Principles of Natural Philosophy*) first published in 1687. This celebrated volume as well as its revised editions in 1713 and 1726, simply called the *Principia* or *Principia Mathematica*, is now universally considered one of the greatest single contributions ever published in the history of physical sciences. In it Newton not only put forward a new theory of how celestial bodies move in space and time, but also developed the complicated mathematics needed to analyze their motion. In addition, he also formulated the laws of motion and the law of universal gravitation according to which each body in the universe was attracted toward every other body by a force that was stronger the more massive the bodies and the closer they were to each other. It was exactly the same force that caused objects to fall to the ground. According to his law, gravity causes the Moon to move in an elliptic orbit around the Earth and causes the Earth and the planets to follow elliptical paths around the Sun. It was the first Newton's book to contain a unified system of physical principles explaining what happens on Earth and in the Universe. Newton's Principia is divided into three volumes, the third volume entitled *On the System of the World* dealt with applications of the fundamental principles formulated in the two proceeding ones to the systematic study of the motion of the heavenly bodies and calculations of the orbits of comets which subsequently verified and extended by a great British astronomer, Edmond Halley. About a third of the third volume is devoted to Newton's theory of the Moon's motion from physical principles. Although Newton's *Principia* was universally acclaimed, but it was severely criticized at that time on philosophical and theological grounds. In response to these criticisms, Newton included a short addendum – the celebrated *General*

Scholium to the second edition of the *Principia* in 1713. So, this addendum was presumably intended to explain many difficult issues in order to prevent further criticism or controversy.

In addition, Newton's other two great books – the *Opticks or a Treatise on the Reflections, Inflections, and Colours of Light*, published in 1704 and the *Arithmetica Universalis*, appeared in 1707 also brought him tremendous reputation in the whole scientific world. With a remarkable contrast to his *Principia*, this treatise retained its real scientific influence and its popular appeal for the non-mathematical reader. In his foreword to the 1931 edition of the *Opticks*, Albert Einstein (1879-1955) described that Newton had early fascination of science with great creative ability as follows:

"... He who has time and tranquility can by reading this book live the wonderful events which the great Newton experienced in his yound days. Nature to him was an open book, whose letters he could read without effort ... In one person he combined the experimenter, the theorist, the mechanic and, not least, the artist in exposition. He stands before us strong, certain and alone; his joy in creation and his minute precision are evident in every word and in every figure."

In his *Opticks*, Newton discovered a corpuscular theory of light that light consists of tiny particles that travel in a straight line through space. He called the particles *corpuscles*. About the same time, Huygens proposed that light consists of waves and suggested the wave theory to explain the nature of light. The corpuscular and wave theories appeared to be completely opposite and scientists continuously argued about them for the next hundred years with no definite resolution. At the beginning of the nineteenth century, a British physician, Thomas Young (1773-1829) experimentally discovered the interference phenomenon of light, and James Clerk Maxwell (1831-1879) discovered a more comprehensive electromagnetic theory describing that electromagnetic waves travel through space at the velocity of light which is considered as an electromagnetic phenomenon. The discoveries of Young and Maxwell for the first time triumphantly established the wave theory of light. Consequently, the corpuscular theory of Newton was not only considered unsuccessful, but it was believed to be wrong throughout the nineteenth century. However, the entire story was completely changed at the beginning of the twentieth century when a great German scientist, Max Planck (1858-1912) discovered *quanta* (or *photons*) and strongly pointed to a particle nature of light. In 1900, Planck proposed a new revolutionary quantum hypothesis for the derivation of the black body radiation that is essentially concerned with thermodynamics of the exchange of energy between radiation and matter. Planck postulated that such energy is emitted only in discrete quantities of magnitude, $E = h\nu$, where h is the Planck constant and ν is the frequency. Indeed, Planck considered the light *quanta* or *photons* as the electromagnetic waves. Based on this revolutionary idea, Planck and Einstein independently derived the celebrated Planck radiation formula for the energy in the form

$$E(\nu)d\nu = \frac{8\pi\nu^2}{c^3} \cdot \frac{\nu^2 d\nu}{\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]}, \tag{14.1.1}$$

where c is the velocity of light, T is the temperature, and k is the Ludwig Boltzmann (1844-1906) constant. This law has an excellent agreement with several experimental findings, and was derived by several authors on the assumption of radiation as electromagnetic waves. Einstein also discovered the *photoelectric effect* by postulating the existence of discrete *quanta* (now called *photons*) of light particles. All these authors were faced with some kind of difficulties, but they have never been able to resolve them.

In 1924, S. N. Bose (1894-1974) was the first to challenge the classical statistical mechanics and totally abandoned Planck's wave aspects of photons. Bose treated radiation as *photon particles* that are indistinguishable identical and massless particles of energy $\varepsilon = h\nu$ and momentum $p = (h\nu/c)$. He then gave an entirely new and novel derivation of the Planck formula (14.1.1) for the energy based on a systematic phase-space argument of statistical mechanics without any assumption of wave aspects of photons. This new *particle concept of radiation* as well as the new derivation of the Planck law is the greatest fundamental discovery of Bose. Both Bose and Einstein totally resuscitated Newton's corpuscular theory of light. By the 1925s, all physicists were prepared to give true recognition of the corpuscular hypothesis and the wave hypothesis in a satisfactory manner as the *wave-particle duality* of light. This remarkable blending of the two fundamental concepts is exactly what Newton proposed in his *Opticks* which became not merely as a historical landmark, but for its living scientific legend.

In summary, Newton described the basic principles of the Solar System and its first physical model. His work on astronomy was remarkably influenced by Edmond Halley through their scientific correspondence and Halley's fateful visit to Cambridge to see Newton in April and again in August of 1684. Halley became very famous in whole Europe for his pioneering work in astronomy and is widely remembered for his famous discovery, universally known as *Halley's Comet* which is fully accepted as a major and permanent member of the Solar System with a period of about 76 years. Among other things, Halley's more refined calculations with mathematical tables took account of the gravitational influence of Jupiter and Saturn and led him to predict that the motion of some comets was periodic and that they moved in highly elongated elliptical orbits rather than parabolic orbits. The next landmark in Newton's academic life was Halley's encouragement to publish the *Principia* as well as Roger Cotes' selection for the editorial job of the *Principia*. Cotes spent a considerable amount of time and energy for his meticulous and skillful editorial work to complete the second revised edition of the *Principia* with significant improvements which was duly published in 1713. On the other hand, Halley not only recognized great contributions of Newton to mathematical sciences and physics, but also provided both editorial and financial help to Newton for publication of the *Principia*. Indeed, Edmond Halley was a true friend and scientific admirer of Newton and persuaded Newton to publish all of his discoveries. Unfortunately, Newton's work was severely criticized by many of his contemporary scientists including Christian Huygens and Robert Hooke.

Based on the major classical works of Kepler, Galileo, Newton and Halley, Euler made remarkable and useful work on astronomy, mechanics, celestial mechanics, many branches of physics, navigation and cartography. This chapter is devoted to major contributions of Euler to astronomy and physics in some detail.

14.2 Euler's Contributions to Astronomy

Euler began to study astronomial problems in his first St. Petersburg period from 1727 to 1741. During this period, he investigated many problems of physics and astronomy – especially celestial mechanics. He published many papers on the theory of both perturbed and unperturbed motion of celestial bodies. He also gave some special attention to calculations of the attraction of an elliptical spheroid, problems of spherical astronomy and astronomical problems related to optics, and other branches of physics. Euler made exceptionally large contributions to these areas. Along with several other great mathematical scientists including J. L. Lagrange, P. S. Laplace, C. F. Gauss, A. C. Clairaut, H. Poincaré, Euler may be considered one of the founding fathers of modern astronomy and celestial mechanics.

Euler's research on unperturbed motion of celestial bodies, and the calculation of orbits were based on the laws of Kepler and those of Newton.

In his studies of the unperturbed motion, Euler considered the heliocentric motion of planets and comets, and obtained an approximate solution of the parabolic motion of a comet. Unfortunately, several astronomers found Euler's graphical solution unsatisfactory because Halley's discovery of comets with elliptic orbits led to the problem of determining orbits of comets with no *priori* assumptions that they must be parabolic. However, Euler made some significant progress to determine orbits of comets from observational data. In 1744, he published research monograph on *Theoria Motuum Planetarium et Cometarum* (*The Theory of Motion of Comets and Planets*), with the solutions of the main problems of theoretical astronomy dealing with the structure, nature, motion and action of comets and planets.

With regard to the theory of perturbed motion of celestial bodies, Euler formulated the perturbation theory in general terms so that it can be used to solve the mathematical problem of dynamic models and particular problems of theoretical astronomy. In order to determine the mass of Halley's Comet, Euler calculated the perturbation of the Earth's orbit caused by the passage of this Comet in 1759. However, his analytical method was unable to produce accurate results that were in excellent agreement with observational data. Since observations revealed no measurable changes in the earth's motion, Euler demonstrated that the masses of the comets are less than planetary masses by several orders of magnitude.

Euler was deeply interested in both the two-body problem (The Earth and the Moon) and the three-body problem (The Sun, Earth and Moon) of the Solar System. He gave an extensive mathematical treatment of the problem of improving approximations of orbits within the framework of the two-body problem and taking perturbations into account. In his *Theoria motuum planetarum et cometarum* published in 1744, Euler gave a complete mathematical treatment of the two-body problem consisting of a planet and the Sun. Another three-body problem in the Solar System dealt with the Sun, Jupiter and Saturn, or any general three-body problem. His mathematical analysis of this problem revealed that neither the observed motion of Jupiter and Saturn nor that of the Moon could be explained completely by the Newton Inverse Square Law. It is worth noting that the Paris Academy of Sciences selected the subject of the three-body problem for its 1748 and 1752 Prize problems. Euler was awarded the prize in both 1748 and 1752 for his research on the irregularities of the orbits of Jupiter and Saturn and for his major contributions to celestial mechanics. Based on the assumption that the orbits of these planets are elliptic in nature so that the position of each is determined by its radius vector and longitude, Euler derived the differential equations for these planets. He then solved these equations using a new method of successive approximations. Since the mutual interaction of any three-body problem perturbs their orbits significantly, the problem became very difficult to make further progress in celestial mechanics. Euler considered special cases of the three-body problem under suitable assumptions and approximations.

In order to study problems of celestial mechanics, Euler expressed Newton's second law of motion as a system of ordinary differential equations

$$(F_x, F_y, F_z) = m\,\frac{d^2}{dt^2}(x, y, z). \tag{14.2.1}$$

For a fixed body of mass M at the origin and a moving body of mass m at (x, y, z), the components of gravitational force are given by

$$(F_x, F_y, F_z) = -\frac{GMm}{r^3}(x, y, z), \tag{14.2.2}$$

where G is the universal gravitational constant, and $r^2 = x^2 + y^2 + r^2$. This gravitational law can easily be extended to the case when both bodies move under their mutual attraction. Based on Newton's universal law of gravitation, Euler first developed his first lunar theory with the aid of his method of variation of orbital parameters. This method is fairly general in the sense that it can not only be applied to the theory of lunar motion, but also to the planetary motion. Euler published his first lunar theory in his celebrated treatise '*Theory of lunar motion*' in 1753. He continued his research for almost the next three decades to make significant improvement of his first lunar theory including the lunar orbit, Moon's position, equations for the Moon's motion, lunar eclipses and the period of revolution of the Moon. In 1772, Euler wrote an extensive and voluminous monograph with a long title *The theory of lunar motion, treated by means of a new method, including astronomical tables from which the moon's position at any time may easily be calculated. An easy composed under the supervision of Leonhard Euler with incredible zeal and untiring labor by three academicians J. A. Euler, W. L. Kraft and A. J. Lexell.* This expanded and most complete second lunar theory dealt with clear and accurate exposition of a theory of motion of celestial bodies. Based on his expanded second lunar theory, Euler completed his new mathematical tables for calculating the position of the Moon that was published from Berlin in 1746. His major works dealt with the two-body problems (Moon - Earth, Earth - Sun, or any two planets), and two three-body problems (Sun - Earth - Moon, and Sun - Jupiter - Saturn) and the computation of the perturbations involved with two or more planets and with all planets combined. He often considered perturbation problems of the planets in conjunction with the Moon or comets.

Euler made a more accurate investigation of the perturbations of the Earth's motion caused by the Moon in 1747, and later in the 1770s his work was devoted to the theory of the perturbations of the earth's motion under the influence of Venus. Subsequently, he published papers under the title: *A new method of producing astronomical tables for the motion of the planets* in 1774 and *A new method of determining the motion of the planets* in 1781. He then used his own lunar method to the study of unperturbed planetary motion using the heliocentric coordinates X, Y of a synodic reference frame for the position of a planet in the plane of the orbit where the axis of abscissas passes through the mean position of the planet. Introducing $X = a(1 + x)$ and $Y = ay$, where a is the major semi axis of the planet's orbit, Euler derived the following results

$$x = e\,P + e^2 Q + e^3 R + \cdots\,, \quad y = e\,p + e^2 q + e^3 r + \cdots\,, \tag{14.2.3}$$

where e is the eccentricity of the orbit and coefficients P, Q, R, $\cdots$ and p, q, r, $\cdots$ are periodic function of the mean anomaly which are determined from the solutions of a certain system of differential equations with constant coefficients.

In one of his memoirs, Euler reported that "... the celebrated three-body problem, which arises from the study of lunar motion, is still too far beyond the power of analysis for one to be able to hope to find a complete solution." And in another place, he wrote: "From this I draw the undeniable conclusion that one cannot hope to solve the general case of the three-body problem while no means is known for solving it even in the case where the three-bodies move along one and the same line." Following this work, Euler obtained solutions for special cases of the one-dimensional three-body problem, now known as "*collinear Lagrangian*" points. In 1766, Euler published an article under the title "On the motion of a body attached to two fixed centers of force". In this work, Euler obtained for the first time a general integral solution of the planar problem of two fixed centers, expressed in terms of elliptic integrals and Jacobi's elliptic functions. A few years later, Lagrange also found the general solution of the spatial problem of two fixed centers, again in terms of elliptic integrals and Jacobi elliptic functions. Obviously, the Euler-Lagrange solution of the problem of two fixed centers is a special case of the Newton three-body problem where a study was made of a passively gravitating body in the field created by two fixed attractive centers.

Both Euler and Clairaut made a serious attempt to obtain exact solutions for the general three body problem and reported mathematical difficulties and then suggested approximate methods. In 1747, Clairaut made first significant progress based on the series solutions of the differential equations. He also applied his results to the motion of Halley's Comet which was observed in 1531, 1607 and 1682. Clairaut calculated the perturbations due to the attraction of two planets, Jupiter and Saturn. It is interesting to note that Euler became interested in the problem of the integrability of the equations arising in the three-body problems in celestial mechanics. In 1763, Euler wrote an article under the title "Remarks on the three-body problems" which played a fundamental role in the development of new method of numerical integration of differential equations in theoretical astronomy. In this paper, Euler discovered one of the *new method* of numerical integration of differential equations of celestial mechanics. This new method is now known as the *Euler–Cole numerical method* at the beginning of the twentieth century as J. D. Cole rediscovered one of the most effective difference methods of integration of differential equations that led to the well-known method of representing solutions in the form of power series in time.

In addition to his work in celestial mechanics, Euler made some contributions to some problems in geodesy and mathematical cartography in both his first and second St. Petersburg periods. While he was in charge of the Geography Department of the St. Petersburg Academy, Euler was actively involved in cartographic research in collaboration with the famous French astronomer and geographer, J. N. Delisle, and, indeed, they were directly responsible for the organization of the St. Petersburg Observatory of the first ever time service in Russia. Both Euler and Delisle were engaged in the development of methods of astronomical observations. Based on their observations for a period of ten years, they computed the instant of true noon. Both were very fascinated by sunspots and they used Delisle's method to compute the trajectories of the sunspots. On the other hand, Euler provided help to Delisle for the determination of the orbits of comets by analytical methods. Euler's contribution to mathematical cartography consisted of a series of three major papers published by the St. Petersburg Academy in 1777 including (i) On the representation of a spherical surface on the plane, (ii) On the geographic projection of the surface of a sphere, and (iii) On Delisle's geographic projection and its use in the general map of the Russian Empire. In collaboration with G. Heinsius on a research project to prepare a map of Russia, Euler then participated directly in the production of the resulting *Geographic Atlas of the Russian Empire* published in 1745. Indeed, three celebrated mathematical scientists including Lambert, Euler and Lagrange laid the modern foundation of the applied science of mathematical cartography and prepared the basic ground for Gauss' work on conformal mappings and differential geometry.

In his 1764 article entitled *considerationes de motu corporum coelestium*, Euler was the first to study the three-body problem in astronomy under certain assumptions and approximations and noted the intractability of the solution of the problem which can be explained by means of his own quote:

"There is no doubt that Kepler discovered the laws according to which celestial bodies move in their paths, and that Newton proved them—to the greatest advantage of astronomy. But this does not mean that the astronomical theory is at the highest level of perfection. We are able to deal completely with Newton's inverse-square law for two bodies. But if a third body is involved, so that each attracts both other bodies, all the arts of analysis are insufficient . Since the solution of the general problem of three bodies appears to be beyond the human powers of the author, he tried to solve the restricted problem in which the mass of the third body is negligible compared to the other two. Possibly, starting from special cases, the road to the solution of the general problem may be found. But even in the case of the restricted problem the solution encounters difficulties so great that the author has to admit to have spent much effort in vain attempts at solution."

On the other hand, based on the Newton's inverse square law, three mathematical scientists including d'Alembert, Clairaut and Euler continued to develop a fairly mathematical lunar theory under certain simplifying assumptions and approximations. However, their works on the lunar theory were somewhat controversial, and so, they raised questions about the validity of Newton's inverse square law. Subsequently, the lunar theory and two three-body problems as stated above had received considerable attention by famous astronomers of that time.

Euler made a lot of correspondence with d'Alembert with frequent disagreement on some issues. However, after d'Alembert visit to Euler in Berlin in 1763, their relation became more cordial. There was a priority dispute between them on the theory of the precession of the equinoxes and nutation of the axis of the Earth.

We conclude this section by adding an interesting quotation of V. K. Abalakin and E. A. Grebenikov from their article on Euler and the Development of Astronomy in Russia published in the MAA (Mathematical

Association of America) Tercentenary Euler Celebration volume entitled *Euler and Modern Science* (2007).

"Euler's correspondence with his contemporaries must be included as part of his priceless creative heritage. In those letters one finds a wealth of fresh scientific questions, solutions of new problems, conclusions concerning a great variety of topics, ranging from the philosophical to the everyday, and, finally, intelligent thoughts of a general nature. Reading these letters, one cannot but be impressed by his philanthropic nature, and by his sincere respect for every correspondent irrespective of title, authority, or social standing. On the other hand, the manner in which he defends his scientific views and results in the face of criticism not always just or well-founded, is instructive and worthy of emulation. Although he always showed respect for the point of view of an appropriate opponent, he did not permit his own take on a subject to be undervalued or derided and refused to compromise his principles. The single and unwavering motive behind his abundant correspondence with contemporary scientists was the attainment and defense of scientific truth. In a large number of the letters there are discussions of astronomical topics. Many of Euler's letters to the mathematician and mechanician P. - L. Maupertuis, and the astronomers N. L. de la Caille, T. Mayer, G. Heinsius, J. N. Delisle, and others, constitute by themselves remarkable discourses on astronomy, cogently and elegantly argued and dealing with the most topical scientific problems of the time, for instance the problem of the earth's shape — then a burning question in all academies — and the related question of the interpretation of the measurements of longitude made on several continents, as well as problems at the juncture of astronomy and mechanics and a great many other exceedingly interesting problems of natural science.

We would like to say in conclusion that the works of Leonhard Euler continue in our day to serve as an almost inexhaustible source of fresh creative ideas, so that studying them today is just as appropriate and useful as it was during the lifetime of the great scientist."

14.3 Euler's Work on Physics

During his years in Berlin, Euler wrote his famous *Letters to a German Princess, Anhalt-Dessaus*, niece of the King of Prussia on different subjects in natural philosophy, astronomy, optics, music, acoustics, mechanics, electricity and magnetism which was one of the most popular science books ever written in the history of sciences. This is an expository survey of general physics and metaphysics. It was translated into eight different languages and became the first encyclopedia of physics in Russia.

It may not be out of place to mention Euler's discovery of the foundation of modern hydrodynamics of inviscid incompressible and compressible fluids. In 1736, he also published his two large volumes, *Mechanica sive motus scientia analytice exposita (Mechanics or the science of motion, expounded analytically).* This two-volume *Mechanica* dealt with a comprehensive treatment of almost all aspects of mechanics including the mechanics of rigid, flexible and elastic bodies as well as fluid mechanics, elasticity, celestial mechanics and ballistics.

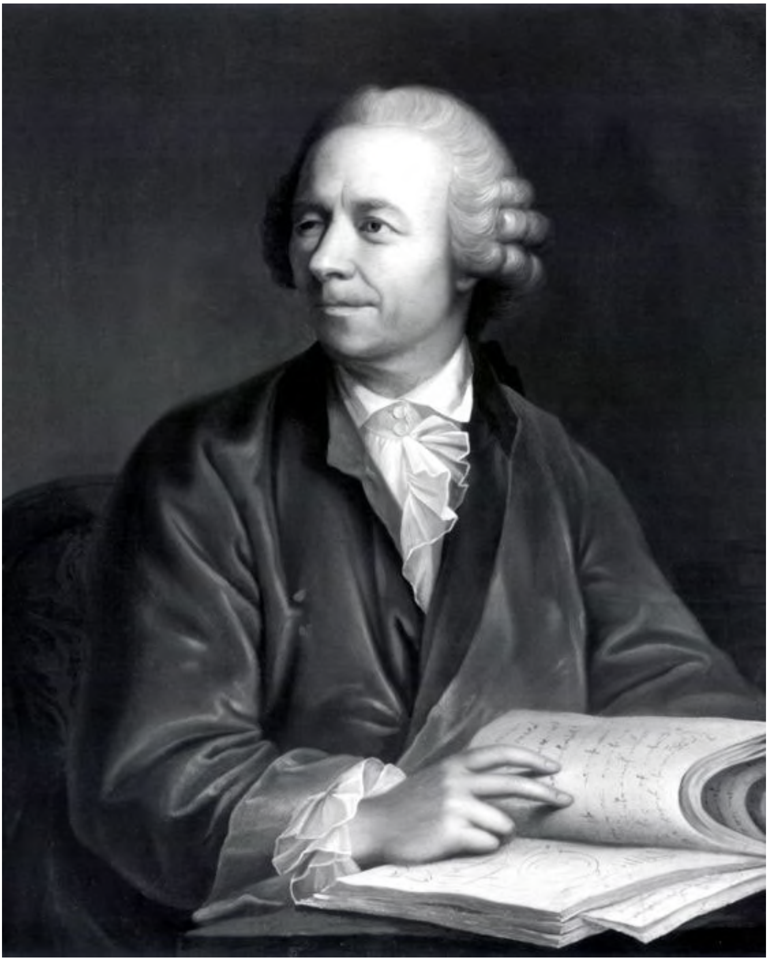
Euler's major contributions to physics are devoted to various topics in physical and geometrical optics, and optical instruments. During 1768-1770, the three volumes of Euler's *Dioptrics* were published. This work dealt with his extensive research in optical sciences and optical instruments including telescopes and microscopes.

In addition, Euler gave a comprehensive treatment of diffraction in the atmosphere. His book *Dioptrics* deals with the determination of the path of a ray of light through a system of diffraction spherical surfaces. In the first approximation, Euler discovered the familiar formulas of elementary optics, and in the second approximation, he took into account the spherical and chromatic aberrations with the spherical aberration errors of the third order. He often discussed the problems of acoustics, optics, electricity and magnetism with the strong indication that they are closely related subjects and therefore, they should receive simultaneous and equal treatment from the mathematical and physical view points.

No doubt, Leonhard Euler was an universal genius and was fully equipped with almost unlimited powers of imagination, intellectual gifts and extraordinary memory. It is again a delight to quote Nikolai Fuss from Eulogy in Memory of Leonhard Euler:

"Knowledge that we call erudition was not inimical to him. He had read all the best Roman writers, knew perfectly the ancient history of mathematics, held in his memory the historical events of all times and peoples, and could without hesitation adduce by way of examples the most trifling of historical events. He knew more about medicine, botany, and chemistry than might be expected of someone who had not worked especially in those sciences."

Finally, Euler laid the mathematical foundations of potential theory, and the theory of shipbuilding based on the principles of hydrostatics. His work on the theory of ships and the motion of a ship culminated in the publication of *Scientia navalis seu tractatus de construendis ac dirigendis navibus* in 1749. In 1773, he published his complete theory of shipbuilding and navigation of ships which became very useful for all who practice seafaring. His monograph entitled, 'An investigation of the physical causes of incoming and outgoing sea-tides' dealt with a dynamical theory of tides and oscillations of bodies of water in an ocean. For this original research work, Euler shared the prize of the Paris Academy of Science in 1740 with Daniel Bernoulli and Colin Maclaurin who had also submitted papers on the similar subject for the prize. He was significantly influenced by the work of great mathematical scientists who preceded him as well as by the remarkable contributions of his contemporaries. It is hoped that enough has been said to give some impression about the topics, variety and depth of Euler's mathematical and physical achievements.



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